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INDIRECT QUANTITATIVE METHODS FOR DETERMINING SOLUTIONS TO ORDI--ETC(U)  
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INDIRECT QUANTITATIVE METHODS FOR  
DETERMINING SOLUTIONS TO ORDINARY  
DIFFERENTIAL EQUATIONS

by

Harold J. Harris  
December 1979

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REPORT DOCUMENTATION PAGE			READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/NR 79-239T	2. GOVT ACCESSION NO. <i>AD-A105380</i>	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (If Different) <i>Indirect Quantitative Methods for Determining Solutions to Ordinary Differential Equations.</i>	5. TYPE OF REPORT & PERIOD COVERED THESIS/DISSERTATION		
6. AUTHOR(s) <i>Capt. Harold J. Harris</i> <i>10</i>	7. CONTRACT OR GRANT NUMBER(s) <i>9 - master's thesis</i>		
8. PERFORMING ORGANIZATION NAME AND ADDRESS AFIT STUDENT AT: California State University, Fresno	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS <i>11 21 Dec 79</i>		
11. CONTROLLING OFFICE NAME AND ADDRESS AFIT/NR WPAFB OH 45433	12. REPORT DATE <i>13. NUMBER OF PAGES 43</i>		
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) <i>12 57</i>	15. SECURITY CLASS. (of this report) UNCLASS		
16. DISTRIBUTION STATEMENT (of this Report) APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) <i>29 SEP 1981</i>			
18. SUPPLEMENTARY NOTES APPROVED FOR PUBLIC RELEASE: IAW AFR 190-17	<i>FREDRIC C. LYNCH</i> FREDRIC C. LYNCH, Major, USAF Director of Public Affairs Air Force Institute of Technology (ATC) Wright-Patterson AFB, OH 45433		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) ATTACHED			

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 68 IS OBSOLETE

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INDIRECT QUANTATIVE METHODS FOR  
DETERMINING SOLUTIONS TO ORDINARY  
DIFFERENTIAL EQUATIONS

Presented to  
The Faculty of the Department of Mathematics  
California State University, Fresno

In Partial Fulfillment  
of the Requirements for the Degree  
Master of Science in Mathematics

by

Harold J. Harris

December 1979

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The subject of differential equation constitutes a large and very important branch of modern mathematics. From the early days of Calculus the subject has been an area of great theoretical research and practical applications, and it continues to be so to date. Therefore, a natural question arises: How does one obtain useful information from a differential equation. The answer is essentially that if it is possible to do so, one solves the differential equation to obtain a solution, if it is not possible, one uses the theory of differential equations to obtain information about the solution. In the following text we will investigate some ordinary differential equations in self-adjoint form and when possible determine the behavior of solutions without actually solving the differential equation.

The following definitions will be used:

SELF-ADJOINT EQUATION is a second order linear differential equation of the form  $[r(x)y']' + p(x)y = 0$ , where  $r(x)$  and  $p(x)$  are continuous and  $r(x) > 0$  on an interval  $a \leq x \leq b$ .

A BOUNDARY-VALUE PROBLEM is a problem which consist of:

1. a second order homogenous linear differential equation of the form  $[r(x)y']' + [p(x) + \lambda q(x)]y = 0$ , where  $p$ ,  $q$ , and  $r$  are real valued functions such that  $q$  has a continuous derivative,  $p$  and  $r$  are continuous,  $q(x) > 0$  and  $r(x) > 0$  for all real  $x$  on an interval  $a \leq x \leq b$ , and  $\lambda$  is a parameter independent of  $x$ .

2. two supplementary conditions

$$A_1y(a) + A_2y'(a) = 0$$

$$B_1y(b) + B_2y'(b) = 0$$

where  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  are real constants such that  $A_1$  and  $A_2$  are not both zero and  $B_1$  and  $B_2$  are not both zero.

A STURM-LIOUVILLE PROBLEM consists of a differential equation of the form  $[r(x)y']' + [p(x) + q(x)]y = 0$ , and boundary conditions

$$A_1y(a) + B_1y'(a) = 0$$

$$A_2y(b) + B_2y'(b) = 0.$$

A SINGULAR STURM-LIOUVILLE EQUATION has any one of the following conditions:

1. the interval is finite and  $p(x)$  vanishes at one or both endpoints,
2. the interval is infinite or semi-infinite.

A REGULAR STURM-LIOUVILLE EQUATION has an interval which is closed and finite, that is, for every real  $x$  the  $a \leq x \leq b$ .

EIGENVALUES are the values of  $\lambda$  in the differential equation

$[r(x)y']' + [p(x) + q(x)]y = 0$ , for which there exist non-trivial solutions of the problem.

EIGENFUNCTIONS are the corresponding nontrivial solutions themselves.

The unique graph of the solutions to an initial value problem in an appropriate region is often referred to as a solution path.

The isolated critical point  $(0,0)$  is called a saddle point if there exists a neighborhood of  $(0,0)$  in which the following conditions hold:

1. There exist two paths which approach and enter  $(0,0)$  from opposite directions as  $t \rightarrow \infty$  and there exist two other paths which approach and enter  $(0,0)$  from different opposite directions as  $t \rightarrow -\infty$ .
2. In each of the four domains between any two of the four paths in (1) there are infinitely many paths which are arbitrarily closed to  $(0,0)$  but which tend away from  $(0,0)$  as  $t \rightarrow +\infty$  and as  $t \rightarrow -\infty$ .

In the study of self-adjoint equations we need a well-known theorem on point sets which is the Bolzano-Wiestrass Theorem. Now suppose  $E$  is a set of points on the  $x$ -axis. A point  $x_0$  is called a limit point of  $E$  if there exist a sequence of distinct points  $x_1, x_2, x_3, \dots$  of  $E$  such that  $\lim_{n \rightarrow \infty} x_n = x_0$ . The Bolzano-Wiestrass Theorem states that every infinite set  $E$  has at least one limit point.

All differential equations can be put in self-adjoint form. In the discussion that follows we will be investigating the zeros of the solutions of self-adjoint equations.

Theorem 1. Let  $f(x)$  be a solution of  $[r(x)y'] + p(x) = 0$  whose first derivative  $f'(x)$  exists and is defined on  $a \leq x \leq b$ . If  $f(x)$  has an infinite number of zeros on  $a \leq x \leq b$  then  $f(x) \equiv 0$  for all  $x$  on  $a \leq x \leq b$ .

Proof: Since  $f(x)$  has an infinite number of zeros on  $[a, b]$  then by the Bolzano-Wiestrass Theorem the set of zeros has a limit point  $x_0$ , where  $x_0 \in [a, b]$ . Thus there exist a sequence  $\{x_n\}$  of zeros which converge to  $x_0$  (with  $x_n \neq x_0$ ). Since  $f(x)$  is continuous then  $\lim_{x \rightarrow x_0} f(x)$ , where  $x \rightarrow x_0$  through any sequence of points

on  $[a, b]$ . Let  $x \rightarrow x_0$  through the sequence of zeros  $\{x_n\}$ . Then  $\lim_{x \rightarrow x_0} f(x) = 0 = f(x_0)$ . Since  $f'(x)$  exists, we have

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

where  $x \rightarrow x_0$  through the sequence  $\{x_n\}$ . Now for such points

$\frac{f(x) - f(x_0)}{x - x_0}$  we have  $f'(x) = 0$ . Therefore by the Uniqueness

Theorem for Differential Equations,  $f(x)$  is a solution of a self-adjoint equation such that  $f(x_0) = f'(x_0) = 0$  for all  $x$  on  $a \leq x \leq b$ . Q.E.D.

ABEL'S FORMULA. Given two solutions  $f(x)$  and  $g(x)$  of a self-adjoint equation on an interval  $a \leq x \leq b$ , then for all  $x$  on  $a \leq x \leq b$  we have  $r(x)[f(x)g'(x) - f'(x)g(x)] = K$ .

Proof: Since  $f(x)$  and  $g(x)$  are solutions of a self-adjoint equation on  $a \leq x \leq b$ , we have

$$[r(x)f'(x)]' + p(x)f(x) = 0 \quad (2)$$

and

$$[r(x)g'(x)]' + p(x)g(x) = 0. \quad (3)$$

Multiplying equation (2) by  $-g(x)$  and equation (3) by  $-f(x)$  and adding the resultant equations we obtain

$$f(x)[r(x)g'(x)]' - g(x)[r(x)f'(x)]' = 0.$$

Integrating from  $a$  to  $x$  and using integration by parts we have

$$f(t)r(t)g'(t)\Big|_a^x - \int_a^x r(t)g'(t)f'(t)dt - g(t)r(t)f'(t)$$

$$+ \int_a^x r(x)f'(t)g'(t)dt = 0$$

which reduces to

$$r(x)[f(x)g'(x) - f'(x)g(x)] = r(a)[f(a)g'(a) - f'(a)g(a)]$$

The right hand member is a constant and by letting  $K$  represent this constant we have  $r(x)f(x)g'(x) - f'(x)g(x) = K$  Q.E.D.

Lemma 1. If two solutions  $f(x)$  and  $g(x)$  of a self-adjoint equation have a common zero then they are linearly dependent. Conversely, if  $f(x)$  and  $g(x)$  are linearly dependent solutions, neither identically zero, then if one of them vanishes at  $x = x_0$  so does the other.

Proof: For the first part we shall apply Abel's Formula.

Let  $x_0 \in [a, b]$  be a common zero of  $f(x)$  and  $g(x)$ . Letting  $x = x_0$  we obtain  $k = 0$ ; hence,  $r(x) [f(x)g'(x) - f'(x)g(x)] = 0$  for all  $x \in [a, b]$ . Since we have assumed  $r(x) > 0$  on  $a \leq x \leq b$  this means that the quantity in the brackets must be zero. But the quantity in the brackets is  $W(f, g)(x)$  which is the Wronskian for  $f$  and  $g$ . By Wronskian definition of linearly dependent, if the determinant

$$W[f(x), g(x)] = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0$$

then the solutions  $f(x)$  and  $g(x)$  are linearly dependent.

Second part. Since  $f(x)$  and  $g(x)$  are linearly dependent on  $a \leq x \leq b$ , then there exist constants  $c_1$  and  $c_2$ , not both zero, such that  $c_1 f(x) + c_2 g(x) = 0$ . By our hypothesis, neither  $f(x)$  or  $g(x)$  is zero. Hence if  $c_1 = 0$  then  $c_2 g(x) = 0$ , but  $g(x) \neq 0$  which means  $c_2$  must be zero. But this is a contradiction to  $c_1$  and  $c_2$  both being zero. Hence  $c_1 \neq 0$  and  $c_2 \neq 0$ . By letting  $x = x_0$  then if  $f(x_0) = 0$  then  $g(x_0) = 0$ . Q.E.D.

Theorem 2. (Sturm Separation Theorem). Let  $f(x)$  and  $g(x)$  be linearly independent solutions of  $[r(x)y']' + p(x)y = 0$  on the interval  $a \leq x \leq b$ , then between any two consecutive zeros of  $f(x)$  there is precisely one zero of  $g(x)$ .

Proof: Let  $x_1$  and  $x_2$  be two consecutive zeros of  $f(x)$  on  $[a, b]$ . Then by lemma 1 we know that  $g(x_1) \neq 0$  and  $g(x_2) \neq 0$ . Assume that  $g(x)$  has no zero on  $x_1 < x < x_2$ . Since the solutions  $f(x)$  and  $g(x)$  have continuous derivatives on  $a, b$  then the quotient  $f(x)/g(x)$  has a continuous derivative on  $x_1 \leq x \leq x_2$ .

Furthermore  $f(x)/g(x) = 0$  at the endpoints of this interval.

Hence by Rolle's Theorem there exist  $\xi$  where  $x_1 < \xi < x_2$  such that

$$\left[ \frac{f(x)}{g(x)} \right]'_{x=\xi} = 0$$

Put  $\left[ \frac{f(x)}{g(x)} \right]' = \frac{W(f, g)(x)}{[g(x)]^2}$  and since  $f(x)$  and  $g(x)$  are linearly independent on  $a \leq x \leq b$ , we have  $\left[ \frac{f(x)}{g(x)} \right]' \neq 0$  on  $x_1 < x < x_2$ . But this is a contradiction. Hence  $g(x)$  has at least one zero on  $x_1 < x < x_2$ .

Now suppose  $g$  has more than one zero in  $x_1 < x < x_2$ . Let  $x_3$  and  $x_4$  be the two consecutive zeros of  $g$ . By interchanging  $f$  and  $g$  in the above discussion, this shows that  $f$  must have at least one zero, say  $x_5$ , in the open interval  $x_3 < x_5 < x_4$ . Then we would have  $x_1 < x_5 < x_2$  meaning that  $x_1$  and  $x_2$  are not consecutive zeros of  $f$ . This is a contradiction to our assumption that  $x_1$  and  $x_2$  were consecutive zeros. Therefore  $g$  has precisely one zero in the interval  $x_1 < x < x_2$ . Q.E.D.

Theorem 3. (Sturm Comparison Theorem) Let  $f(x)$  be a solution of  $[r(x)y']' + p(x)y = 0$  and  $g(x)$  be a solution of  $r(x)y' + p_1(x)y = 0$ . Let  $r(x) > 0$  and have a continuous derivative and let  $p(x)$  and  $p_1(x)$  be continuous and  $p_1(x) > p(x)$ . If  $x_1$  and  $x_2$  are consecutive zeros of  $f(x)$  on  $[a, b]$ , then  $g(x)$  has at least one zero at some point on the interval  $x_1 < x < x_2$ .

Proof: Assume  $g(x)$  does not have a zero on the interval  $x_1 < x < x_2$ . By our hypothesis we have

$$[r(x)y']' + p(x)y = 0$$

$$[r(x)y']' + p_1(x)y = 0$$

for all  $x \in [a, b]$ . Now we multiply the first equation by  $g(x)$  and the second equation by  $f(x)$  and subtract. The resultant equation is

$$r(x)[f(x)f'(x)]' - f(x)[r(x)g'(x)]' = [p_1(x) - p(x)]f(x)g(x)$$

The left hand side of the equation yields

$$r(x)[f(x)f'(x)]' - f(x)[r(x)g'(x)]' = \{r(x)[f'(x)g(x) - f(x)g'(x)]\}'.$$

Therefore

$$\{r(x)[f'(x)g(x) - f(x)g'(x)]\}' = [p_1(x) - p(x)]f(x)g(x).$$

Integrating from  $x_1$  to  $x_2$  we have

$$\int_{x_1}^{x_2} \{r(x)[f'(x)g(x) - f(x)g'(x)]\}' dx = \int_{x_1}^{x_2} [p_1(x) - p(x)]f(x)g(x)dx$$

or

$$r(x)[f'(x)g(x) - f(x)g'(x)] \Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} [p_1(x) - p(x)]f(x)g(x)dx$$

Since  $f(x_1) = f(x_2) = 0$  then the above equation becomes

$$r(x_2)f'(x_2)g(x_2) - r(x_1)f'(x_1)g(x_1) = \int_{x_1}^{x_2} [p_1(x) - p(x)]f(x)g(x)dx \quad (4)$$

By our hypothesis  $r(x_2) > 0$ . Now since  $f(x_2) = 0$  and  $f(x) > 0$  on  $x_1 < x < x_2$ , we have  $f'(x_2) < 0$ . Also since  $g(x) > 0$  then  $g(x_2) \geq 0$ .

Hence  $r(x_2)f'(x_2)g(x_2) \leq 0$ . In a similiar manner, we have

$r(x_1)f'(x_1)g(x_1) \geq 0$ . Therefore the left hand side of equation (4) is not positive. Also, by our hypothesis  $p_1(x) > p(x)$  implies that  $p_1(x) - p(x) > 0$ . So the right hand side of equation (4) is positive. This is a contradiction to the assumption that  $g(x)$  does not have a zero on the interval  $x_1 < x < x_2$ . Q.E.D.

Theorem 4. (Osgood Theorem) Suppose  $p(x) > 0$  and  $p'(x) \geq 0$  on the interval  $0 \leq x < \infty$  and let  $y(x)$  be an arbitrary solution of  $y'' + p(x)y = 0$ . If  $x=a$  and  $x=b$  are two consecutive zeros of  $y'(x)$  then  $|y(b)| \leq |y(a)|$ .

Proof: Suppose  $p'(x) \neq 0$  on the interval  $a \leq x \leq b$ , then  $y(x)$  is constant and  $|y(b)| = |y(a)|$ . If  $p'(x) \neq 0$  and  $y(x)$  is a solution of  $y'' + p(x)y = 0$  then the equation becomes

$$y''(x) + p(x)y(x) = 0 \quad (5)$$

Multiplying equation (5) by  $2y'(x)$  and integrating the resultant equation over the interval  $a, b$  we obtain

$$y'^2(x) \int_a^b + \int_a^b p(x)[Y^2(x)]' dx = 0$$

The first term on the left is zero, therefore our equation becomes

$$\int_a^b p(x) y^2(x) ' dx = 0.$$

Using integration by parts we have

$$\int_a^b p(x) y^2(x) ' dx = p(b)y^2(b) - p(a)y^2(a) - \int_a^b p'(x)y^2(x)dx = 0$$

$$\text{or } p(b)y^2(b) - p(a)y^2(a) = \int_a^b p'(x)y^2(x)dx \quad (6)$$

Suppose  $y^2(b) > y^2(a)$ , then  $p(b)y^2(b) - p(a)y^2(b)$

$$p(b)y^2(b) - p(a)y^2(a),$$

then  $p(b)y^2(b) - p(a)y^2(b)$  is greater than both the left and right members of equation (6). It follows that

$$\begin{aligned} p(b)y^2(b) - p(a)y^2(b) &> p(b)y^2(b) - p(a)y^2(a) \\ &\sim p(a)y^2(b) > - p(a)y^2(a) \\ p(a)y^2(b) &< p(a)y^2(a) \end{aligned}$$

But this contradicts our assumption  $y^2(b) > y^2(a)$ . Therefore  $y^2(b) \leq y^2(a)$  and  $|y(b)| \leq |y(a)|$  on the interval  $0 \leq x < \infty$ . Q.E.D.

Theorem 5. Let  $r(x)$  be positive and suppose that  $r(x)$  and  $p(x)$  are continuous on the interval  $0 < x < \infty$ . If

$$\int_1^\infty \frac{1}{r(x)} dx = +\infty, \text{ and} \quad (7)$$

$$\int_1^\infty p(x)dx = +\infty \quad (8)$$

Then every solution  $y(x)$  of equation (1) vanishes infinitely often on the interval  $1 < x < \infty$ . Similarly, in the intervals

$$\int_c^1 \frac{1}{r(x)} dx = +\infty, \text{ and} \quad (9)$$

$$\int_1^c p(x)dx = +\infty \quad (10)$$

then every solution  $y(x)$  of equation (1) vanishes infinitely often on the interval  $0 < x < 1$ .

Theorem 6. Every nontrivial solution of  $[r(x)y']' + p(x)=0$  has at most a finite number of zeros on the interval  $a \leq x < \infty$ , if

$$\int_a^\infty \frac{1}{r(x)} dx < +\infty, \text{ and} \quad (11)$$

$$\left| \int_a^x p(x)dx \right| < M \quad (12)$$

where  $M$  is any positive constant.

Theorem 7. (Lienard-Levinson-Smith Theorem) A differential equation  $\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$

where  $f$ ,  $g$ , and  $F$  defined by  $F(x) = \int_0^x f(u)du$  and  $G$  defined by  $G(x) = \int_0^x g(u)du$  are real functions having the following properties:

1.  $f$  is even and continuous for all  $x$ .

2. there exists a number  $x_0 > 0$  such that  $F(x) < 0$  for  $0 < x < x_0$ ,  $F(x) > 0$  and monotonic increasing for  $x > x_0$ . Furthermore we have  $F(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

3.  $g$  is odd, has a derivative for all  $x$ , and  $g$  is such that  $g(x) > 0$  for  $x \geq 0$ .

4.  $G(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Then the differential equation  $\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$

has a unique nontrivial periodic solution. In other words the equivalent autonomous system  $\frac{dx}{dt} = y$ ,  $\frac{dy}{dt} = -f(x)y - g(x)$

has a unique closed path in the xy plane.

**Proof:** Recall an autonomous system is a system in which the independent variable appears only in the differentials on the left members and not in the functions on the right. In the above system,  $t$  is the independent variable. Considering the differential equation  $\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$  where  $f(x) = E(x^2 - 1)$ ,  $E$  constant, and  $g(x) = x$  we have the well known Van der Pol equation

$$\frac{d^2x}{dt^2} + E(x^2 - 1)\frac{dx}{dt} + x = 0.$$

It follows that  $F(x) = \int_0^x f(u)du$   
 $= \int_0^x E(u^2 - 1)du$   
 $= E\left(\frac{u^3}{3} - u\right) \Big|_0^x$   
 $= E\left(\frac{x^3}{3} - x\right)$

and  $G(x) = \int_0^x g(u)du$   
 $= \int_0^x u du = \frac{x^2}{2}.$

Checking to see if our properties hold we have

1. since  $f(-x) = E(x^2 - 1) = f(x)$ , then the function  $f$  is even and clearly  $f$  is continuous for all  $x$ .

2.  $F(x) = E(x^3/3 - x) = x^2/3(x^2 - x)$ . Here

$$F(\sqrt{3}) = 0$$

$$F(x) < 0 \text{ if } x < \sqrt{3}$$

$$F(x) > 0 \text{ if } x > \sqrt{3}$$

$F(x)$  is monotone increasing and  $F \rightarrow \infty$

as  $x \rightarrow \infty$ .

3. Since  $g(-x) = -x = -g(x)$ , it follows that the function  $g(x)$  is odd. Also since  $g'(x) = 1$ , then the derivative of  $g$  is continuous for all  $x$ . It is obviously true that  $g(x) > 0$  for  $x > 0$ .

Therefore the Van der Pol equation  $\frac{d^2y}{dt^2} + E(x^2 - 1)\frac{dy}{dx} + x = 0$

satisfies all the properties of the Lienard-Levinson-Smith Theorem and the equation has a unique nontrivial periodic solution.

Furthermore the equivalent autonomous system

$$\frac{dx}{dt} = y \quad \frac{dy}{dt} = E(1-x^2)y - x$$

has a unique closed path. Reducing the above autonomous system to a differential equation of the paths, we have,

$$\frac{dy}{dt} \frac{dx}{dt} = E(1-x^2)y - x \quad y$$

Using the chain rule we have

$$\frac{dy}{dx} = \frac{E(1-x^2)}{y} - x \quad \text{Q.E.D.}$$

Now we will discuss how to put differential equations in self-adjoint form. Given a linear differential equation of the second order

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad (13)$$

where  $a(x) > 0$  and  $a(x)$ ,  $b(x)$ , and  $c(x)$  are continuous on the interval  $[a, b]$ , this equation can be put in self-adjoint form by the following procedure: First multiply both members of the equation by the function

$$\frac{1}{a(x)} e^{\int_a^x \left[ \frac{b(x)}{a(x)} \right] dx}$$

Next set  $r(x) = e^{\int_a^x \left[ \frac{b(x)}{a(x)} \right] dx}$

and  $p(x) = \frac{c(x)}{a(x)} e^{\int_a^x \left[ \frac{b(x)}{a(x)} \right] dx}$

The self-adjoint equation will be  $[r(x)y']' + p(x)y = 0$ . A necessary and sufficient condition that equation (13) be self-adjoint is that  $[a(x)]' = b(x)$  on  $a \leq x \leq b$ .

Using the above procedure, we shall put the Bessel, Hermite, and Laguerre equations in self-adjoint form. Consider the Bessel equation  $x^2y'' + xy' + (x^2 - n^2)y = 0$ , where  $n$  is a constant. It follows that

$$\frac{1}{a(x)} e^{\int_a^x \left[ \frac{b(x)}{a(x)} \right] dx} = \frac{1}{x^2} e^{\int_x^a \frac{1}{x} dx} = \frac{1}{x}$$

Multiplying both members of the Bessel equation by  $1/x$ , we have

$$xy'' + y' + [(x^2 - n^2)/x]y = 0,$$

Therefore  $r(x) = x$  and  $p(x) = \frac{x^2 - n^2}{x}$ , and the self-adjoint form of

the Bessel equation is  $xy'' + (x - \frac{n^2}{x})y = 0$  (14)

Consider the Hermite equation  $y'' - 2xy' + 2ny = 0$  where  $n$  is a constant, we have

$$\frac{1}{a(x)} e^{\int_a^x \left[ \frac{b(x)}{a(x)} \right] dx} = e^{-\int x dx} = e^{-x^2}$$

Multiplying each term by  $e^{-x^2}$  we find  $r(x)=e^{-x^2}$  and  $p(x)=2ne^{-x^2}$ .

Therefore the self-adjoint form of the Hermit equation is

$$[e^{-x^2} y']' + 2ne^{-x^2} y = 0. \quad (15)$$

Consider the Laguerre equation  $xy'' + (1-x)y' + ny = 0$ , where  $n$  is a constant. We have

$$\frac{1}{a(x)} e^{\int \frac{b(x)}{a(x)} dx} = \frac{1}{x} e^{\int \frac{(1-x)}{x} dx} = e^{-x}.$$

Multiplying each term by  $e^{-x}$  we obtain

$$e^{-x} xy'' + (1-x)e^{-x} y' + ne^{-x} y = 0.$$

Therefore the self-adjoint form of the Laguerre equation is

$$[xe^{-x} y']' + ne^{-x} y = 0.$$

Now we are going to give examples of how to investigate oscillation of solutions of differential equations without actually solving the differential equation.

The Sturm Separation Theorem essentially says that the zeros of one of two linearly independent solutions of a self-adjoint equation separate the zeros of the other solution. That is, the number of zeros on an interval of a self-adjoint equation is about the same as the number of zeros of the other solution. For example, we can show that between any two consecutive zeros of  $\sin x + \sqrt{2}\cos x$  there is one zero of  $\sin x - \cos x$ . We know by the Sturm Separation Theorem that the zeros of one linearly independent solution of a self-adjoint equation separates the zeros of the solutions of the other self-adjoint equation. First we shall construct our equations in self-adjoint form and apply

Sturm's Separation Theorem. In the first function we let

$$v = \sin x + \sqrt{2}\cos x, \text{ then}$$

$$v'' = -\sin x - \sqrt{2}\cos x, \text{ therefore}$$

$v'' + v = 0$  is our self-adjoint equation with  $r(x) = p(x) = 1$ .

In the second function we let

$$z = \sin x - \cos x, \text{ then}$$

$$z'' = -\sin x + \cos x.$$

Here we have another self-adjoint equation  $z'' + z = 0$ . Thus  $v(x)$  and  $z(x)$  are two linear independent solutions of  $y'' + y = 0$ .

By applying Sturm's Separation Theorem we know between any two consecutive zeros of  $\sin x + \sqrt{2}\cos x$  there is one zero of  $\sin x - \cos x$ .

Given two self-adjoint equations, what can one say about their solutions? We find that the Sturm Comparison Theorem can best answer this question. The theorem compares the rates of oscillation of solutions of the two equations.

We now consider the two differential equations

$x^2y'' + xy' + y = 0$  and  $y'' + y = 0$  and investigate which possesses the more rapidly oscillating solution on the interval  $(1, \infty)$ .

We know by Theorem 3 that if two equations are in self-adjoint form and  $p_1(x) > p(x)$  then the solutions of the self-adjoint equation which contains  $p_1(x)$  will oscillate more rapidly. The second equation is already in self-adjoint so all that remains is to put the first equation in self-adjoint form and apply Theorem 3.

Putting the first equation in self-adjoint we  $[xy']' + \frac{1}{x}y = 0$ .

Hence our two equations become

$[xy']' + \frac{1}{x}y = 0$  and  $y'' + y = 0$ . Since  $\frac{1}{x} < 1$  for all  $x$  in  $1 < x < \infty$ ,

then by the Sturm Comparison Theorem the solutions of  $y'' + y = 0$  will oscillate more rapidly than the solutions of  $[xy']' + \frac{1}{x}y = 0$ .

Suppose we are given two differential equations

$$x^2y'' + xy' + (x^2 - 1)y = 0, \text{ and} \quad (16)$$

$$xz'' + z' + xz = 0 \quad (17)$$

with nontrivial solutions  $u(x)$  and  $z(x)$  respectively, and both equations vanish at  $x=1$ . Now we determine which equation will vanish first after  $x=1$ . This problem can best be solved by first putting the two equations in self-adjoint form and applying Theorem 3. We know by Theorem 3 if  $u(x)$  and  $z(x)$  are solutions of two self-adjoint equations then the differential equation with the larger second term will oscillate more frequently. Our objective, therefore, would be to reconstruct the two equations and compare the coefficient of the second term of each equation. Both equations vanish at  $x=1$ , so we must find which equation oscillates with the highest frequency beginning after  $x=1$ . The respective self-adjoint forms of equations (16) and (17) are

$$\left[\frac{1}{x}y'\right]' + (x-1/x)y = 0, \text{ and}$$

$$z'' + \frac{1}{x}z = 0.$$

Considering only the second term of both equations for  $x > 1$  we have  $\frac{1}{x} < x-1/x$ . Therefore by Sturm's Comparison Theorem between any two consecutive zeros of  $z'' + (1/x)z = 0$  there is one zero of  $\left[\frac{1}{x}y'\right]' + (x-1/x)y = 0$ . Since solutions of equations (16) and (17)

both have a common zero at  $x=1$ , then between  $x=1$  and the next zero of  $z'' + (1/x)z = 0$  there is one zero of equation (16). Therefore solutions of equation (16) oscillates with higher frequency than solutions of equation (17), and  $x^2y'' + xy' + (x^2 - ) = 0$  will vanish first after  $x=1$ .

In our discussion thus far we have considered the case where  $p(x) > 0$  and have seen that a nontrivial solution of  $[r(x)y']' + p(x)y = 0$  can have infinitely many zeros on the interval  $[a, b]$ . One may ask the question, what can be said about the nontrivial solutions if  $p(x) \leq 0$ . In the following discussion we shall see that if  $p(x) \leq 0$  then no nontrivial solution of  $[r(x)y']' + p(x)y = 0$  can have more than one zero on  $[a, b]$ .

Let  $u(x)$  be a nontrivial solution of  $r(x)y' + p(x)y = 0$ , and let  $x_0$  be a zero of  $u(x)$  so that  $u(x_0) = 0$ . Since  $u(x)$  is a nontrivial solution, then  $u'(x) \neq 0$ , therefore we have two cases,  $u'(x_0) > 0$  or  $u'(x_0) < 0$ .

Case 1. Suppose  $u'(x_0) > 0$ , then  $u(x)$  is positive over some interval to the right of  $x_0$ . By the hypothesis, the equation is

$$[r(x)y']' + p(x)y = 0, \text{ or}$$

$$[r(x)y'(x)]' + p(x)y(x) = 0.$$

Using variable substitution we have

$$[r(x)u'(x)]' + p(x)u(x) = 0, \text{ or}$$

$$[r(x)u'(x)]' = -p(x)u(x).$$

We are given  $p(x) \leq 0$ . If  $p(x) = 0$  then  $[r(x)u'(x)] = 0$  and we have the desired zero. If  $p(x) < 0$  then  $[r(x)u'(x)]'$  is a positive function on the same interval. This means that the slope  $u'(x)$  is increasing so  $u(x)$  cannot have a zero to the right of  $x_0$ . By

analogous reasoning,  $u(x)$  cannot have a zero to the left of  $x_0$ .

Case 2. Suppose  $u'(x) < 0$  then  $r(x)u'(x)$  is a negative function. This means that the slope  $u'(x)$  is a decreasing function and  $u(x)$  cannot have a zero to the left of  $x_0$ . Similarly  $u(x)$  cannot have a zero to the right of  $x_0$ .

Therefore, if  $p(x) \leq 0$  then a nontrivial solution of  $[r(x)y']' + p(x)y = 0$  can have no more than one zero on  $[a, b]$ .

Now consider the differential equation  $y'' + p(x)y = 0$ , where  $p$  is continuous and  $p(x) > 0$  on  $a \leq x \leq b$ . Let  $p_m$  denote the minimum value of  $p(x)$ . We will show that if  $p_m > \frac{k^2\pi^2}{(b-a)^2}$  then

every solution of  $y'' + p(x)y = 0$  has at least  $k$  zeros on  $a \leq x \leq b$ .

First we consider the two equations

$$y'' + p(x)y = 0, \text{ and} \quad (18)$$

$$y'' + \frac{k^2\pi^2}{(b-a)^2} y = 0. \quad (19)$$

We will examine the solutions of equation (19) and apply Theorem 3. A solution of equation (18) is  $\sin x$  and a solution of equation (19) is  $\sin \left(\frac{k}{b-a}x\right)$  where  $k$  is a multiple of  $b-a$ . By

assumption  $p(x) > \frac{k^2\pi^2}{(b-a)^2}$ , so by Theorem 3 the solutions of equation (18) oscillates with higher frequency than the solutions of equation (19). The solution of equation (19) is  $\sin \frac{k\pi}{b-a} x$  where

$k$  is a multiple of  $b-a$ . Therefore  $y'' + \frac{k^2\pi^2}{(b-a)^2} y = 0$  has at least

$k$  zeros. Since  $p(x) > \frac{k^2\pi^2}{(b-a)^2}$  then by Tehorem 3, between any

two zeros of equation (19) there is at least one zero of equation

(18). Since equation (19) has at least  $k$  zeros therefore  $y'' + p(x)y = 0$  contains at least  $k$  zeros.

The Sturm Comparison Theorem is very helpful in providing information about equations of the form  $y'' + p(x)y = 0$ . Consider the equation  $y'' + p(x)y = 0$  where  $p$  is continuous on  $a \leq x \leq b$  such that  $0 < m < p(x) < M$ . Let  $u(x)$  be a solution of  $y'' + p(x)y = 0$  having consecutive zeros at  $x_1$  and  $x_2$  (where  $a \leq x_1 < x_2 \leq b$ ). Then  $\frac{\pi}{\sqrt{M}} < x_2 - x_1 < \frac{\pi}{\sqrt{m}}$ .

Consider the following equation  $y'' + My = 0$  where  $v$  is a solution such that  $v(x_1) = 0$ . Then  $y = \frac{1}{\sqrt{M}} \sin(x - x_1)\sqrt{M}$ . Since  $M > p(x)$  then by Theorem 3 there will be at least one zero of equation  $y'' + My = 0$  between any two consecutive zeros of  $y'' + p(x)y = 0$ , that is, there will be a zero of  $y'' + My = 0$  between any two zeros  $x_1$  and  $x_2$ . Since  $x_1$  is a zero of  $y'' + p(x)y = 0$  then the zero of  $y'' + My = 0$  will be at  $x_1 + \frac{\pi}{\sqrt{M}}$  and  $x_2 > x_1 + \frac{\pi}{\sqrt{M}}$ ,

hence  $x_2 - x_1 > \frac{\pi}{\sqrt{M}}$ . Now consider  $y'' + my = 0$  where  $w$  is a solution such that  $w(x_2) = 0$ . Then  $y = \frac{1}{\sqrt{m}} \sin(x_2 - x)\sqrt{m}$ . Since  $p(x) > m$ , then by Theorem 3 there is at least one zero of  $y'' + p(x)y = 0$  between any two consecutive zeros of  $y'' + my = 0$ . By assumption  $x_1$  and  $x_2$  are two consecutive zeros of  $y'' + p(x)y = 0$ , therefore a zero of  $y'' + p(x)y = 0$  will be at  $x_1$  and a zero will be at  $x_1 + \frac{\pi}{\sqrt{m}}$  where

$x_1 + \frac{\pi}{\sqrt{m}} > x_2$ . Hence  $x_2 < x_1 + \frac{\pi}{\sqrt{m}}$ , and  $x_2 - x_1 < \frac{\pi}{\sqrt{m}}$ , so that

$$\frac{P}{\sqrt{m}} < x_2 - x_1 < \frac{\pi}{\sqrt{m}} .$$

Applications of the Sturm Comparison Theorem can also be made to the Bessel equation for specific values of the constant.

For example let us examine the Bessel equation for oscillation of its solutions. Let  $y_p(x)$  be a nontrivial solution of the Bessel equation on the positive axis. If  $0 \leq p < \frac{1}{2}$  then every interval of length  $\pi$  contains at least one zero of  $y_p(x)$ . If  $p = \frac{1}{2}$  then the distance between successive zeros of  $y_p(x)$  is exactly  $\pi$  and if  $p > \frac{1}{2}$  then every interval of length  $\pi$  contains at most one zero of  $y_p(x)$ .

By Theorem 3, if  $y(x)$  and  $z(x)$  are two nontrivial solutions of  $y'' + p_1(x)y=0$  and  $y'' + p_2(x)y=0$  respectively, and if  $p_1(x) > p_2(x)$ , then between two successive zeros of  $z(x)$  there is at least one zero of  $y(x)$ . Furthermore, given a differential equation  $y'' + p(x)y' + q(x)y=0$  it can be put in self-adjoint form  $z'' + q_1(x)z=0$  just by a simple change of the dependent variable. One refers to the equation  $y'' + p(x)y' + q(x)y=0$  as the standard form of the equation, and  $z'' + q_1(x)z$  as the normal form.

In solving this problem we take the Bessel equation in its standard form and put it in its normal form. Then take the different values of  $p$ , substitute them into the equation in its normal form, and apply Theorem 3 by comparing the coefficient of the second term.

For equation  $z'' + z=0$  a solution  $u(x)$  is  $\sin x$ . Since  $\sin x$  has vanishing points at  $\sin n\pi$  where  $n=0, 1, 2, \dots$ , then

between any two successive zeros of  $\sin x$ , the length of the interval is  $\pi$ .

Consider the Bessel equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \quad (20)$$

where  $p$  is a parameter. By making the transformation

$y = u(x)x^{-\frac{1}{2}}$ , this reduces equation (20) to

$$u'' + \left[ 1 + \frac{(1-4p^2)}{4x^2} \right] u = 0. \quad (21)$$

Case 1.  $0 \leq p < \frac{1}{2}$ . If  $p=0$  then equation (21) becomes

$$u'' + \left[ 1 + \frac{1}{4x^2} \right] u = 0 \quad (22)$$

Therefore if  $y_p(x)$  is a solution of equation (21) then  $y_p(x)$  is a solution of  $u'' + \left[ 1 + \frac{1}{4x^2} \right] u = 0$ . Let  $z(x)$  be a solution of

$z'' + z = 0$ . The coefficient of  $u$  in equation (22) is greater than the coefficient of  $z$  in  $z'' + z = 0$ . Therefore by Theorem 3 between any two successive zeros of  $z'' + z = 0$  there is at least one zero of a solution of equation (22). We previously noted that any two successive zeros of  $z(x)$  are at intervals  $\pi$  distance apart, therefore in any interval of length  $\pi$  there is at least one zero of  $y_p(x)$ . Now consider the condition  $0 < p < \frac{1}{2}$ . Using the coefficient of  $u$  in equation (21) and substituting the value of the parameter  $p$  we have

$$1 + \frac{(1-4p^2)}{4x^2} > 1 + \frac{1-4(\frac{1}{2})^2}{4x^2}$$

$$\frac{1-4p^2}{4x^2} > \frac{1-4(\frac{1}{2})}{4x^2}$$

$$\frac{1-4p^2}{4x^2} > 0$$

The coefficient of  $u$  in equation (21) is  $1 + \frac{(1-4p^2)}{4x^2}$ , where

$\frac{(1-4p^2)}{4x^2} > 0$ , and the coefficient of  $z$  in  $z'' + z = 0$  is 1. It

follows that  $1 + \frac{(1-4p^2)}{4x^2} > 1$ , and by Theorem 3 between any two

successive zeros of a solution of  $z'' + z = 0$  there is at least one zero of equation (21). But the zeros in  $z'' + z = 0$  are  $\pi$  distance apart; therefore, for  $0 < p < \frac{1}{2}$  in equation (21) there is at least one zero in any interval of length  $\pi$ .

Case 2. For  $p = \frac{1}{2}$  we have

$$u'' + \left[1 + \frac{1-4p^2}{4x^2}\right]u = 0$$

$$u'' + \left[1 + \frac{1-0}{4x^2}\right]u = 0$$

$$u'' + u = 0.$$

The successive zeros of  $u'' + u = 0$  occur at the same intervals as the successive zeros of  $z'' + z = 0$ . Therefore, when  $p = \frac{1}{2}$  the successive zeros of equation (21) occur at intervals of length  $\pi$ .

Case 3.  $P > \frac{1}{2}$ . Considering only the coefficient of  $u$  in equation (21) we have  $1 + \frac{1-4p^2}{4x^2} < 1 + \frac{1-4(\frac{1}{2})^2}{4x^2}$

$$\frac{1-4p^2}{4x^2} < \frac{1-4(\frac{1}{2})^2}{4x^2}$$

$$\frac{1-4p^2}{4x^2} < 0.$$

Here the coefficient of  $u$  is negative. In an earlier example

we showed that if  $p(x) < 0$  in equation (1) then no nontrivial solution can have more than one zero on the interval  $a \leq x \leq b$ .

Since  $1 + \left(\frac{1-4p^2}{4x^2}\right) > 1$  for  $p > \frac{1}{2}$ , then by Theorem 3 between any two

successive zeros of  $z'' + z = 0$  there is at least one zero of equation (21). But for  $p > \frac{1}{2}$ , equation (21) has at most one zero on any interval. The successive zeros of  $z'' + z = 0$  are  $\pi$  distance apart. Therefore for  $p > \frac{1}{2}$  on any interval  $\pi$  distance apart, there is at most one zero of  $u'' + \left[1 + \frac{(1-4p^2)}{4x^2}\right]u = 0$ .

From our applications of Theorem 3 (Sturm Comparison Theorem) we know if we are given a differential equation

$$y'' + p(x)y = 0 \quad (23)$$

where  $p(x)$  is positive on the interval  $0 \leq x < \infty$ , then  $p(x) > p(0) > 0$  and every solution of equation (23) vanishes infinitely often on the interval. What can one say about the amplitudes of the oscillation. Do they increase, decrease, or remain the same as  $x$  increases on the interval. W. Osgood investigated this problem and developed the Osgood Theorem (Theorem 4). He discovered that under the conditions stated above the amplitudes of the oscillations of solutions of equation (23) never increase as  $x$  increase on the interval  $0 \leq x < \infty$ .

Now let us test the Bessel equation  $x^2y'' + xy' + (x^2 - n^2)y = 0$  for oscillations on the interval  $1 < x < \infty$ . We know from our previous discussion that the self-adjoint form of this equation is

$$[xy']' + \left(x - \frac{n}{x}\right)y = 0$$

We also know by Theorem if  $\int_1^\infty \frac{1}{x} dx = +\infty$ , then every solution  $y(x)$

vanishes infinitely often on the interval  $1 < x < \infty$ . But  $\int_1^\infty \frac{dx}{x}$

does not exist and the test fails. We also know from our previous investigation of the Bessel equation that for an interval of length  $\pi$  and certain values of  $n$  we do have oscillations. Making the transformation  $y = x^{-\frac{1}{2}}u$ , the Bessel equation becomes

$$u'' + \left[ \frac{1-4n^2-1}{4x^2} \right] u = 0$$

Now divide the  $x$ -axis into intervals of length  $\pi$ . If  $0 < n < \frac{1}{2}$ , then every interval has at least one zero of a particular solution  $y(x)$ . Therefore every solution  $y(x)$  of the Bessel equation in the normal form

$$u'' + \left[ \frac{1-4n^2-1}{4x^2} \right] u = 0 ,$$

with  $0 < n < \frac{1}{2}$ , vanishes infinitely often on the interval  $1 < x < \infty$  with the length of each sub-interval equal to  $\pi$ .

Our work on oscillations has prepared us for a brief study of another area: The Sturm-Liouville Problems. Our first concern is a study of a special type of two point boundary-value problem. There are two important cases of the Sturm-Liouville problem, those in which the supplementary conditions are either of the form

$$\begin{aligned} y(a) &= 0, \quad y(b) = 0, \quad \text{or} \\ y'(a) &= 0, \quad y'(b) = 0 . \end{aligned} \tag{24}$$

Consider the following boundary-value problem:

$$(xy')' + (2x^2 + \lambda x^3)y = 0 , \tag{25}$$

with conditions  $3y(1) + 4y'(1) = 0$ , and

$$5y(1) - 3y'(1) = 0 .$$

Equation (25) is of the form  $[r(x)y']' + [p(x)+q(x)]y = 0$

where  $r(x)=x$ ,  $p(x)=2x^2$ , and  $q(x)=\lambda x^3$ ; and the conditions are of the form

$$A_1 y(a) + B_1 y'(a) = 0, \text{ and}$$
$$A_2 y(b) + B_2 y'(b) = 0.$$

To solve this problem we must find a function  $f$  which satisfies both the differential equation and the supplementary conditions. One solution to any problem of this type is the trivial solution  $w$  such that  $w(x)=0$  for all values of  $x$ . Even though this is a solution to our problem, one must admit this trivial solution is not very useful. Therefore we focus our attention on the search for nontrivial solutions. During our investigation we shall see that the existence of such nontrivial solutions depends upon the value of the parameter  $\lambda$  in  $[r(x)y']' + [p(x)+\lambda q(x)]y = 0$ . To illustrate this, let us consider the Sturm-Liouville problem

$$\begin{aligned} y'' + y &= 0 \\ y(0) &= 0, \quad y(\pi) = 0. \end{aligned} \tag{26}$$

We have three cases:  $\lambda=0$ ,  $\lambda<0$ , or  $\lambda>0$ . In each case we shall first find the general solution of the differential equation and then attempt to determine the two arbitrary constants in the solution so that the supplementary conditions will also be satisfied.

Case 1.  $\lambda=0$ . For this case the differential equation  $y''+y=0$  reduces to  $y''=0$ . Hence, the general solution is  $y=c_1+c_2x$ . Applying the first supplementary condition  $y(0)=0$ , we have  $c_1+c_2\pi=0$ . Since  $c_1=0$  then  $c_2$  must be zero. Therefore in order for our solution to satisfy our conditions, we must have  $c_1=c_2=0$ . Our solution then becomes  $y(x)=0$  for all values of  $x$ . Therefore for the parameter  $\lambda=0$ , the only solution of  $y''+\lambda y=0$

is the trivial solution.

Case 2.  $\lambda < 0$ . The characteristic equation of equation (26) is  $m^2 + \lambda = 0$ . It follows that  $m^2 = -\lambda$

$$m = \pm \sqrt{-\lambda}$$

Since  $\lambda < 0$ , the roots are real and unequal. Let  $a = \sqrt{-\lambda}$ , then the general solution of equation (26) is

$$y = c_1 e^{ax} + c_2 e^{-ax}.$$

Applying the first condition  $y(0)=0$  we have,  $c_1 + c_2 = 0$ . By the second condition  $y(\pi)=0$  we have,  $c_1 e^{a\pi} + c_2 e^{-a\pi} = 0$ . In order for the solution  $y=c_1 e^{ax} + c_2 e^{-ax}$  to satisfy the conditions  $y(0)=0$  and  $y(\pi)=0$ , the constants  $c_1$  and  $c_2$  must satisfy the system of equations  $c_1 + c_2 = 0$  and  $c_1 e^{a\pi} + c_2 e^{-a\pi} = 0$ . Obviously  $c_1 = c_2 = 0$  is a solution of this system but this is only the trivial solution. Now we find the non zero values of  $c_1$  and  $c_2$ . Recall that a necessary and sufficient condition that  $n$  homogeneous linear algebraic equations in  $n$  unknown have a nontrivial solution is that the determinants of the coefficients of the system be equal to zero. Therefore we have

$$\begin{vmatrix} 1 & 1 \\ e^{a\pi} & e^{-a\pi} \end{vmatrix} = 0$$

$$e^{-a\pi} - e^{a\pi} = 0$$

$$e^{-a\pi} = e^{a\pi}$$

For the equality to hold,  $a$  must equal zero. That is, for the nontrivial function  $y=c_1 e^{a\pi} + c_2 e^{-a\pi}$  to satisfy the conditions  $y(0)=0$  and  $y(\pi)=0$ , we must have  $a=0$ . But  $a=\sqrt{-\lambda}$  which implies that  $\lambda = 0$ . By assumption  $\lambda \neq 0$ , therefore for this case there are no nontrivial solutions.

Case 3.  $\lambda > 0$ . Since  $\lambda > 0$ , then the roots  $\pm\sqrt{\lambda}$  of the characteristic equation are the conjugate complex numbers  $\pm\sqrt{\lambda}i$ . Therefore the general solution of equation (26) is

$$y = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x.$$

Applying the first condition  $y(0)=0$ , we have  $c_1 \sin 0 + c_2 \cos 0 = 0$ . Since  $\cos 0 = 1$  then  $c_2 = 0$ . Applying the second condition  $y(\pi) = 0$ , we have  $c_1 \sin \sqrt{\lambda}\pi + c_2 \cos \sqrt{\lambda}\pi = 0$ . Since  $c_2 = 0$  the above equation reduces immediately to  $c_1 \sin \sqrt{\lambda}\pi = 0$ . This equation can be solved in either of two ways; set  $c_1 = 0$  or set  $\sin \sqrt{\lambda}\pi = 0$ . However, if  $c_1 = 0$ , we have already established that  $c_2 = 0$ , then the solution reduces to the trivial solution,  $c_1$  must not equal zero. So we must have  $\sin \sqrt{\lambda}\pi = 0$ . But  $\sin k\pi = 0$  only if  $k$  is a positive integer. In order for equation (26) to have a nontrivial solution of the form

$$y = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$$

satisfying the conditions  $y(0) = 0$  and  $y(\pi) = 0$  we must have  $\lambda = n^2$ , where  $n = 1, 2, 3, \dots$ . In summary, if  $\lambda \leq 0$  then the Sturm-Liouville problem consisting of  $y'' + \lambda y = 0$  and conditions  $y(0) = 0$ ,  $y(\pi) = 0$  does not have a nontrivial solution. However, if  $\lambda > 0$  then a nontrivial solution can exist only if  $\lambda$  is one of the values given by  $\lambda = n^2$  where  $n = 1, 2, 3, \dots$ . Furthermore, from the general solution  $y = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$  we find the nontrivial solutions corresponding to  $\lambda = n^2$  are given by

$y = c_n \sin nx$  where  $c_n$  is an arbitrary nonzero constant. In other words, the functions defined by  $c_1 \sin x$ ,  $c_2 \sin 2x$ ,  $c_3 \sin 3x$ , ..., where  $c_1, c_2, c_3, \dots$ , are arbitrary non zero constants, are nontrivial solutions of the problem.

A Sturm-Liouville equation can have an interval which is finite, semi-infinite, or infinite. By the definitions at the beginning of the text, the equation is classified either as a regular or singular Sturm-Liouville equation. Two immediate examples of singular Sturm-Liouville equations are the Laguerre and Bessel equations. Now consider the boundary-value problem  $y'' + \lambda y = 0$ , with supplementary conditions  $y(0) = 0$  and  $y(\infty) = 0$ . This equation is a Sturm-Liouville equation with  $r(x) = 1$ ,  $q(x) = 1$ , and  $p(x) = 0$ . The interval is finite and the functions  $r(x)$  and  $q(x)$  are positive and continuous, therefore the equation is a regular Sturm-Liouville equation.

The investigation of  $y'' + \lambda y = 0$  and supplementary conditions reveal that the values of  $\lambda$  for which there exist nontrivial solutions are  $\lambda = n^2$  where  $n = 1, 2, 3, \dots$ . By definition 6, these values are called eigenvalues of the problem. By definition 7, the eigenfunction of the problem are the corresponding nontrivial solutions  $y = c_n \sin nx$  where  $n = 1, 2, 3, \dots$ , and the  $c_n$  are arbitrary nonzero constants. Eigenvalues and eigenfunctions play an important role in many equations. One such equation is the Schrödinger wave equation for the harmonic oscillator in quantum mechanics. This equation is

$$\frac{d^2w}{dx^2} + \frac{8\pi^2 m(E - \frac{1}{2}kx^2)}{\hbar^2} w = 0 \quad (27)$$

where  $E$  is the total energy,  $\hbar$  is a constant, and the solutions  $w(x)$  are known as Schrödinger wave functions. Using the equation  $k = 4\pi^2 m v^2$  to eliminate the constant, the equation becomes

$$\frac{d^2w}{dx^2} + \frac{8\pi^2 m}{h^2} (E - 2\pi^2 mv^2 x^2) w = 0 .$$

The physically admissible solutions of this equation are those satisfying the conditions  $w \rightarrow 0$  as  $|x| \rightarrow \infty$  and (28)

$$\int_{-\infty}^{\infty} |w|^2 dx = 1.$$

These solutions of the Schrödinger wave functions are called the eigenfunctions and exist only when  $E$  has certain special values called eigenvalues. If we change the independent variable to

$$u = 2\pi \sqrt{\frac{vm}{x}} x , \text{ or}$$

$$\frac{uh}{4\pi^2 vm} = x^2 ,$$

then equation (28) becomes

$$\frac{d^2w}{du^2} + \frac{4^2 vm}{h^2} \left( E - \frac{u^2 h}{2} \right) w = 0 , \text{ and}$$

$$\frac{d^2w}{du^2} + \frac{2}{vh} \left( E - \frac{u^2 vh}{2} \right) w = 0 ,$$

$$\frac{d^2w}{du^2} + \left( \frac{2E}{vh} - u^2 \right) w = 0 . \quad (29)$$

Equation (29) is a Sturm-Liouville equation with  $\lambda = \frac{2E}{vh} - u^2$ .

The solutions of equation (29) satisfy  $w \rightarrow 0$  as  $|u| \rightarrow \infty$  and

$$\int_{-\infty}^{\infty} |w|^2 du = 2\sqrt{\frac{vm}{h}} .$$

Equation (29), except for the notation has exactly the form of another Sturm-Liouville equation

$$\frac{d^2z}{dx^2} + (2p + 1 - x^2) z = 0 , \quad (30)$$

where  $p$  is a constant. Physicists are only interested in solutions

of this equation that approach zero as  $|x| \rightarrow \infty$ , therefore we shall only consider those solutions. Now we take equation (30) and transform it into a well known equation, the Hermite equation. To simplify equation (30) a new independent variable  $t = ye^{-\frac{1}{2}x^2}$  is introduced. This transforms equation (29) into the equation  $y''' - 2xy' + 2py = 0$  which is the Hermite equation with the self-adjoint form discussed earlier as  $e^{-x^2} y''' + \lambda e^{-x^2} y = 0$ . Recall the Hermite equation has a two form recursion formula

$$a_{n+2} = \frac{-2(p-n)}{(n+1)(n+2)} a_n$$

and the formula generates two independent series solutions:

$$y_1(x) = 1 - \frac{px^2}{2!} + \frac{2^2 p(p-2)x^4}{4!} - \dots, \text{ and}$$

$$y_2(x) = x - \frac{2(p-1)x^3}{3!} + \frac{2^3(p-1)(p-3)x^5}{5!} - \dots.$$

Now  $y_1(x)e^{-\frac{1}{2}x^2} \rightarrow 0$  as  $|x| \rightarrow \infty$  if and only if the series  $y_1(x)$  terminates, that is, if and only if the parameter has one of the values  $0, 2, 4, \dots$ .

**Proof:** IF PART. Using L'Hospital's rule we have

$$\lim_{x \rightarrow \infty} \frac{d/dx y_1(x)}{\frac{d}{dx} e^{\frac{1}{2}x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{(-2px + 2^2 4p(p-2)x^3 - 2^3 6p(p-2)(p-4)x^5 + \dots)}{xe^{\frac{1}{2}x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{(-2px + 2^2 p(p-2)x^3 - 2^3 p(p-2)(p-4)x^5 + \dots)}{xe^{\frac{1}{2}x^2}}.$$

Since the denominator goes to infinity faster than the numerator, then the equation goes to zero as  $|x| \rightarrow \infty$ .

ONLY IF PART. This portion of the proof will be shown using the concept of contrapositive. Assume  $p \neq 0, 2, 4, \dots$  and show

$\frac{y_1(x)}{e^{\frac{1}{2}x^2}}$  does not approach zero.  $y_1(x)$  has the form

$$y_1(x) = \sum a_{2n} x^{2n}$$

with its coefficients determined by  $a_{n+2} = \frac{2(p-n)}{(n+1)(n+2)} a_n$ , and the condition  $a_0 = 1$ . Let  $e^{\frac{1}{2}x^2} = \sum b_{2n} x^{2n}$

where  $b_{2n} = \frac{1}{2^n n!}$ , therefore

$$\frac{y_1(x)}{e^{\frac{1}{2}x^2}} = \frac{a_{2n} x^{2n}}{b_{2n} x^{2n}} = \frac{a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2n} x^{2n} + \dots}{b_0 + b_2 x^2 + b_4 x^4 + \dots + b_{2n} x^{2n} + \dots}$$

To show  $\frac{y_1(x)}{e^{\frac{1}{2}x^2}}$  does not approach zero as  $x \rightarrow \infty$ , it would be

$$\frac{y_1(x)}{e^{\frac{1}{2}x^2}}$$

sufficient to show  $a_{2n} > b_{2n}$  if  $n$  is large enough. Observe that

$$\frac{a_{2n+2}}{a_{2n}} = \frac{-2(p-2n)}{(2n+1)(2n+2)} \quad , \text{ and}$$

$$\frac{b_{2n+2}}{b_n} = \frac{1}{2(n+1)} \quad .$$

Therefore

$$\begin{aligned} \frac{a_{2n+2}/a_{2n}}{b_{2n+2}/b_n} &= \frac{-2(p-2n)}{(2n+1)(2n+2)} \cdot \frac{2(n+1)}{1} \\ &= -\frac{2(p-2n)}{2n+1} \quad . \end{aligned}$$

For  $n$  sufficiently large we have,  $-\frac{2(p-2n)}{2n+1} \rightarrow 2$ . This implies that

$$\frac{a_{2n+2}}{a_{2n}} \cdot \frac{b_{2n}}{b_{2n+2}} > 3/2 \quad , \text{ and}$$

$$\frac{a_{2n+2}}{b_{2n+2}} > \frac{3}{2} \frac{a_{2n}}{b_{2n}} \quad \text{for all sufficiently}$$

large n's. If N is one of these n's, then repeated applications of the above inequality shows that

$$\frac{a_{2N+1}}{b_{2N+1}} > \left(\frac{3}{2}\right)^k \cdot \frac{a_{2N}}{b_{2N}} > 1$$

for all sufficiently large k's. Therefore

$$\frac{a_{2n}}{b_{2n}} > 1, \text{ or } a_{2n} > b_{2n}$$

if n is large enough. Thus  $\frac{y_1(x)}{e^{\frac{1}{2}x^2}}$  does not approach zero as

$|x| \rightarrow \infty$ . Q.E.D.

By a similar argument it can be shown that  $y_2(x)e^{-\frac{1}{2}x^2} \rightarrow 0$  as  $|x| \rightarrow \infty$ , with parameters  $p=1, 3, 5, \dots$ . Therefore our desired solutions are constant multiples of Hermite polynomials  $H_0(x)$ ,  $H_1(x)$ ,  $H_2(x)$ , ... . It has been shown that the Schrödinger equation can be put in the form  $\frac{d^2z}{dx^2} + (2p + 1 - x^2)z = 0$ ,

which can be transformed into a Hermite equation that has solutions satisfying

$$ye^{\frac{1}{2}x^2} \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Hence the Schrödinger equation has solutions satisfying  $w \rightarrow 0$  as  $|u| \rightarrow \infty$  if and only if  $\frac{2E}{hv} = 2p + 1$  or  $E = hv(n + \frac{1}{2})$  where n is non-negative. Therefore we conclude that solutions of

$$\frac{d^2w}{du^2} + \left(\frac{2E}{hv} - u^2\right)w = 0$$

have solutions of the form  $w = ce^{-\frac{1}{2}u^2} H_n(u)$  where  $c$  is a constant.

Now the second condition  $\int_{-\infty}^{\infty} |w|^2 du = 2\pi \sqrt{\frac{m}{h}}$  must be examined.

Substituting  $w = ce^{-\frac{1}{2}u^2} H_n(u)$  we have  $\int_{-\infty}^{\infty} c^2 e^{-u^2} [H_n(u)]^2 du$ .

This integral can be solved using the orthogonality property when  $n=m$  and an application of Rodrigues' formula. This formula for the Hermite polynomials states

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

Therefore

$$\int_{-\infty}^{\infty} c^2 e^{-u^2} [H_n(u)]^2 du = c^2 \int_{-\infty}^{\infty} e^{-u^2} H_n(u) H_n(u) du.$$

Applying Rodrigues' formula, we have

$$\begin{aligned} c^2 \int_{-\infty}^{\infty} e^{-u^2} H_n(u) H_n(u) du &= c^2 (-1)^n \int_{-\infty}^{\infty} e^{-u^2} e^{u^2} \frac{d^n}{dx^n} e^{-u^2} H_n(u) du \\ &= c^2 (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} e^{-u^2} H_n(u) du. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} w = H_n(u) &\quad , \quad dv = \frac{d^n}{du^n} e^{-u^2} du, \text{ and} \\ dw = H_n'(u) &\quad , \quad v = \frac{d^{n-1}}{du^{n-1}} e^{-u^2}. \end{aligned}$$

Note that  $wv = d^{n-1} e^{-u^2} H_n(u)$  is the product of  $e^{-u^2}$  and a polynomial and vanishes at both limits. Therefore

$$c^2 \int_{-\infty}^{\infty} e^{-u^2} [H_n(u)]^2 du = (-1)^{n+1} \int_{-\infty}^{\infty} H_n'(u) \frac{d^{n-1}}{du^{n-1}} e^{-u^2} du$$

$$= (-1)^{n+2} \int_{-\infty}^{\infty} H_n^{(n)}(u) \frac{du}{u^{n-2}} e^{-u^2} du$$

$$= (-1)^{2n} \int_{-\infty}^{\infty} H_n^{(n)}(u) e^{-u^2} du$$

The term containing the highest power of  $u$  is  $2^n u^n$ , therefore

$$H_n^{(n)}(u) = 2^n n! \text{ and the last integral becomes}$$

$$2^n n! \int_{-\infty}^{\infty} e^{-u^2} du = 2^n 2^n n! \int_0^{\infty} e^{-u^2} du$$

Since  $\int_0^{\infty} e^{-u^2} du$  is a gamma function, we have

$$2^n n! \int_{-\infty}^{\infty} e^{-u^2} du = (2^n n!) 2 \frac{\sqrt{\pi}}{2} = 2^n n! \sqrt{\pi}.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-u^2} [H_n(u)]^2 du = 2^n n! \sqrt{\pi},$$

and the second condition becomes

$$\int_{-\infty}^{\infty} |w|^2 du = 2\pi \sqrt{\frac{vm}{h}} = \pi \sqrt{\frac{4vm}{h}}, \text{ and}$$

$$c^2 \int_{-\infty}^{\infty} e^{-u^2} [H_n(u)]^2 du = \pi \sqrt{\frac{4vm}{h}},$$

$$c^2 2^n n! \sqrt{\pi} = \pi \sqrt{\frac{4vm}{h}},$$

$$c^2 = \frac{\sqrt{\pi}}{2^n n!} \sqrt{\frac{4vm}{h}}, \text{ so}$$

$$c = \left[ \frac{4vm}{2^{2n}(n!)^2 h} \right]^{\frac{1}{2}}.$$

Therefore the Schrödinger equation  $\frac{d^2w}{du^2} + \left(\frac{2E}{\hbar^2} - u^2\right)w = 0$  with conditions  $w \geq 0$  as  $|x| \rightarrow \infty$  and  $\int_{-\infty}^{\infty} |w|^2 dx = 1$  is of the form of a Sturm-Liouville equation. The nontrivial solutions exist if and only if the eigenvalue  $\lambda$  is  $\lambda = E = \hbar v(n + \frac{1}{2})$  where  $n \geq 0$ . The non-trivial solutions corresponding to the values of  $\lambda$  are the eigenfunctions

$$w = \left[ \frac{4 \sqrt{v}}{2^{2n}(n!)^2 \hbar} \right]^{\frac{1}{4}} e^{-\frac{1}{2}u^2} H_n(u)$$

The concept of quantitatively analysing differential equations can be applied to understanding the movement of a swinging pendulum. For a simple pendulum one measures the position of the swinging arm by an angular coordinate  $\theta$ . We can learn a good deal about the motion of a pendulum if we convert the equation of motion into a system and discuss its phase portrait and stability of some of its solutions. It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases, we assume the law of conservation of energy, namely, the sum of the kinetic and potential energy to be constant. A system of this kind is said to be conservative. The simplest conservative system consists of a mass  $m$  attached to a spring and moving in a straight line through a vacuum. If we let  $x$  represent the displacement of  $m$  from its equilibrium position, and let the restoring force exerted on  $m$  by the string be  $-kx$  where  $k > 0$ , then the equation of motion is

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (31)$$

A spring of this kind is called a linear spring because the restoring force is a linear function of  $x$ . If  $m$  moves through a resisting medium, and the resistance (or damping force) exerted on  $m$  is  $-c(dx/dt)$  where  $c > 0$ , then the equation of motion of this non-conservative system is  $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0$ . Here we have

linear damping because the damping force is a linear function of  $\frac{dx}{dt}$ . If  $f$  and  $g$  are arbitrary functions with the property that  $f(0)=0$  and  $g(0)=0$ , then the general equation

$$m \frac{d^2x}{dt^2} + g\left(\frac{dx}{dt}\right) + f(x) = 0$$

can be interpreted as the equation of motion of a mass  $m$  under the action of a restoring force  $-f(x)$  and a damping force  $-g(dx/dt)$ . In general these forces are nonlinear and equation (31) can be regarded as the basic equation of nonlinear mechanics. In our discussion we shall consider the special case of a nonlinear conservation system described by the equation

$$m \frac{d^2x}{dt^2} + f(x) = 0, \quad (32)$$

Where the damping force is zero and there is consequently no dissipation of energy. Equation (32) is equivalent to the nonlinear autonomous system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)/m. \quad (33)$$

Eliminating  $dt$ , we obtain the differential equation of the paths defined by solutions of (33) in the  $x,y$  phase plane,

$$\frac{dy}{dx} = -f(x)/my \quad (34)$$

Separating the variables in equation (34) we obtain

$$mydy = -f(x)dx \quad (35)$$

Suppose  $x=x_0$  and  $y=y_0$  when  $t=t_0$ , then by integrating equation(35) from  $t_0$  to  $t$ , we obtain

$$\begin{aligned} \frac{1}{2}my^2 - \frac{1}{2}my_0^2 &= - \int_{x_0}^x f(x)dx, \quad \text{or} \\ \frac{1}{2}my^2 + \int_c^x f(x)dx &= \frac{1}{2}my_0^2 + \int_c^{x_0} f(x)dx. \end{aligned} \quad (36)$$

Now  $\frac{1}{2}my^2 = \frac{1}{2}m(dx/dt)^2$  is the kinetic energy of the system and

$$v(x) = - \int_0^x f(x)dx$$

is the potential energy. Thus equation (36) takes the form of

$$\frac{1}{2}my^2 + v(x) = E, \quad (37)$$

where  $E=\frac{1}{2}my_0^2 + v(x_0)$  is the constant total energy of the system.

Since equation (36) was obtained by integrating equation (35), we see that equation (37) gives the family of paths in the  $x,y$  phase plane. For a given value of  $E$ , the path given by equation (37) is a curve of constant energy in this plane. In other words, along a particular path the total energy of the system is a constant; this expresses the law of conservation of energy. The critical points of the autonomous system (33) are the points with coordinates  $(x_c, 0)$ , where  $x_c$  are the roots of the equation  $f(x)=0$ . These are equilibrium points of the dynamical system. From the differential equation (35) we see that the paths cross the  $x$ -axis at right angles and have horizontal tangents along the lines  $x=x_c$ . Equation (37) also tells us that the paths are symmetrical

with respect to the  $x$ -axis. Solving equation (37) for  $y$ , we find

$$y = \pm \sqrt{\frac{2}{m}} \sqrt{E - v(x)} .$$

Now let us consider a simple pendulum which is composed of a mass  $m$  at the end of a straight wire of negligible mass and length  $n$ . Suppose this pendulum is suspended from a fixed point and is free to vibrate in a vertical plane. Let  $x$  denote the angle which the straight wire makes with the vertical ray downward at time  $t$ . Assume there is no air resistance and the only forces acting on the mass are the tension in the wire and the force due to gravity. The differential equation for the displacement of the pendulum is

$$\frac{d^2x}{dt^2} + k \sin x = 0 , \quad (38)$$

where  $k$  is a positive constant. The above equation is in the form of equation (32), where  $m=1$  and  $f(x)=k \sin x$ . The autonomous system equivalent to equation (38) is

$$\frac{dx}{dt} = y , \quad \frac{dy}{dt} = - k \sin x .$$

The critical points of this system are points  $(x_c, 0)$ , where  $x_c$  are the roots of  $\sin x=0$ . Thus the critical points are an infinite set of points  $(n\pi, 0)$  where  $n=0, \pm 1, \pm 2, \dots$ . Hence the differential equation of the paths in the  $x, y$  phase plane is

$$\frac{dy}{dx} = - \frac{\sin x}{y} .$$

By variable separation we have

$$y dy = - \sin x dx$$

To the equation of the family of paths is

$$\frac{1}{2}y^2 + (k - k\cos x) = E . \quad (39)$$

Equation (39) is of the form of equation (37) where  $m=1$ ,  $\frac{1}{2}y^2$  is the kinetic energy, and  $v(x) = \int_0^x f(x)dx = k - k\cos x$  is the potential energy. Now we shall construct the  $xy$  phase plane diagram by first constructing the curve  $z=v(x)$  and the lines  $z=E$  for various values  $E \geq 0$  (see Figure 1, where  $z=E=2k$ ). From this plane the values  $E-v(x)$  can be read on the graphs and can be plotted in the  $xy$  phase plane directly below the value of  $y$  as

$$\begin{aligned} y &= \pm \sqrt{2[E-v(x)]} \\ &= \pm \sqrt{2[E-(k-k\cos x)]} \end{aligned}$$

Using the formula  $v(x)=k-k\cos x$  for the potential energy, then

$$v'(x)=\sin x \quad \text{and } v''(x)=\cos x .$$

However,  $v'(x)=0$  at  $x=n\pi$  where  $n=0, \pm 1, \pm 2, \dots$ . Observe that  $v''(n\pi)>0$  if  $n$  is even and  $v''(n\pi)<0$  if  $n$  is odd. Therefore the potential energy  $v$  has a relative minimum at  $x=2n\pi$  for every even integer  $2n$  (where  $n=0, \pm 1, \pm 2, \dots$ ) and a relative maximum at  $x=(2n+1)\pi$  for every odd integer  $(2n+1)$ , where  $n=0, \pm 1, \pm 2, \dots$ . Hence the critical points  $(2n\pi, 0)$  where  $n=0, \pm 1, \pm 2, \dots$  are centers and are stable; and critical points  $[(2n+1)\pi, 0]$  (where  $n=0, \pm 1, \pm 2, \dots$ ) are saddle points and unstable.

(Note: The diameter of a closed curve  $c$  is called the maximum of the distance  $(A, B)$  between the points  $A$  and  $B$  on the curve  $c$  for all possible pairs of points  $A$  and  $B$  on  $c$ . Consider Figure 3, the critical point  $(0, 0)$  is called a saddle point and such a point may be characterized as follows:

1. It is approached and entered by two half-line paths (AO and BO) as  $t \rightarrow \infty$ . These two paths forming the geometric curve AB.
2. It is approached and entered by two half-line paths (CO and DO) as  $t \rightarrow \infty$ . These two paths forming the geometric curve CD.
3. Between the four half-line paths described in (1) and (2) there are four domains  $R_1, R_2, R_3, R_4$ , each containing an infinite family of semi-hyperbolic-like paths which do not approach 0 as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$  but which become asymptotic to one of the four half-line paths as  $t \rightarrow +\infty$  and as  $t \rightarrow -\infty$ .)

Using Figure 2 we can see that if the total energy E is less than 2, then the corresponding paths are closed. Each of these paths surrounds one of the centers  $(2n\pi, 0)$ . Physically all of these stable centers correspond to exactly one physical state, namely, the pendulum at rest with the bob in its lowest position (the stable equilibrium position). Therefore each closed path about a center corresponds to a periodic back-and-forth oscillation motion of the pendulum bob about its lowest position. On the other hand, if the total energy is greater than 2, then the corresponding paths are not closed. Clearly  $x \rightarrow +\infty$  as  $t \rightarrow +\infty$  if  $y = \frac{dx}{dy} > 0$ , and  $x \rightarrow -\infty$  as  $t \rightarrow -\infty$  if  $y = \frac{dx}{dy} < 0$ . Thus the motion corresponding to such a path does not define x as a periodic function of t. Nevertheless, physically, the corresponding motion is periodic. To see this observe the following: All the unstable

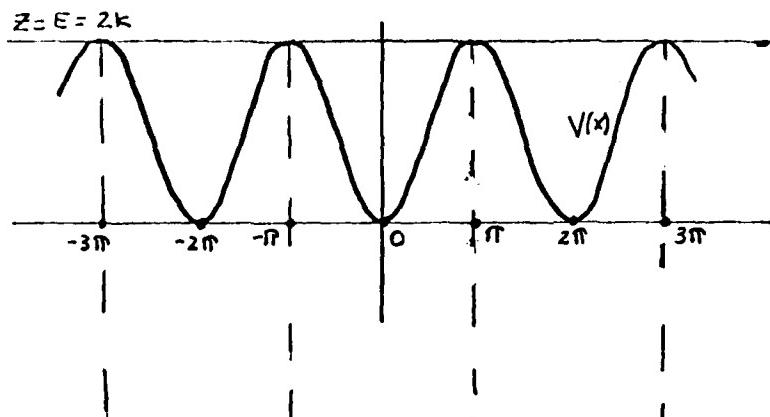


Figure 1

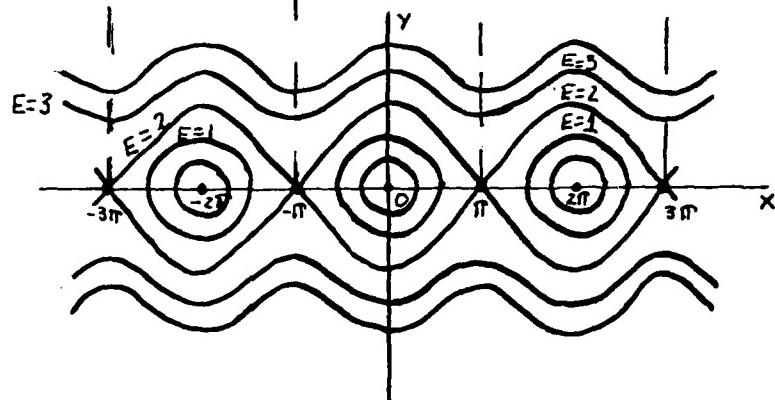


Figure 2

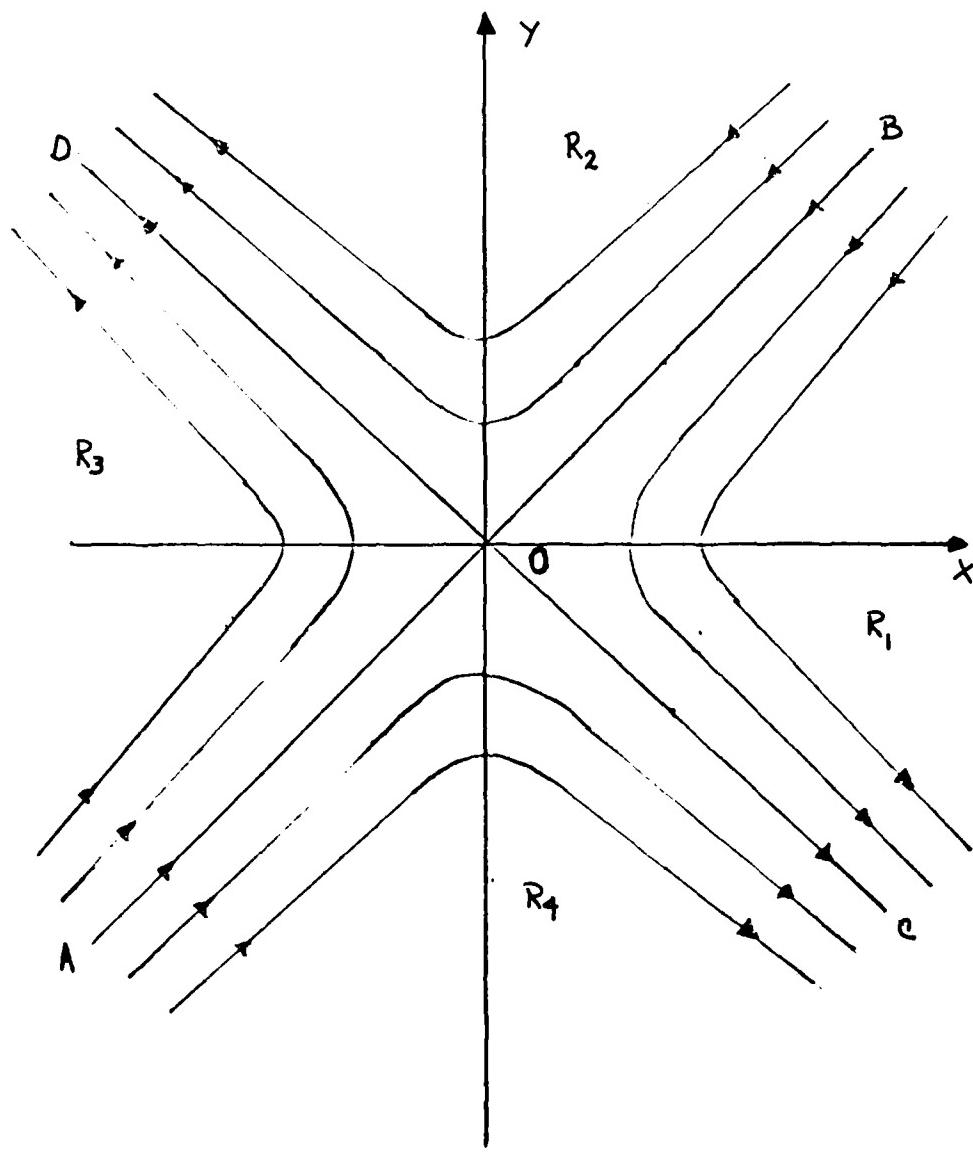


Figure 3

centers  $((2n+1), 0)$  correspond to exactly one physical state, namely, the pendulum at rest with the bob in its highest position; and each of the nonclosed paths corresponds to a physical rotating motion of the pendulum about its pivot point. If the total energy value is  $E=2$ , this separates the back-and-forth motion from the rotational motion. The paths that correspond to  $E=2$  are the ones which enter the saddle points as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ .

Now we turn our discussion to oscillations of second order nonlinear equations. Recall an  $n^{\text{th}}$ -order ordinary equation in the unknown  $y$  and independent variable  $x$  is nonlinear if it can not be put in the form

$$b_n(x) \frac{d^n y}{dx^n} + b_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + b_1(x) \frac{dy}{dx} + b_0(x)y = g(x).$$

The well known Van der Pol equation is a second order nonlinear equation. This equation is

$$\frac{d^2 x}{dt^2} + E(x^2 - 1) \frac{dx}{dt} + x = 0 ,$$

where  $E$  is a positive constant. We shall investigate the Van der Pol equation and see if there exist periodic solutions. Theorem 7 can be used to determine if the Van der Pol equation has periodic solutions providing all of the requirements can be met. Checking the properties of the theorem, we have

1. Since  $f(-x) = E(x^2 - 1) = f(x)$ , then the function  $f$  is even.

Clearly  $f$  is continuous for all  $x$ .

2. For  $F(x)$ , we have  $F(x) = E(\frac{x^3}{3} - x)$ . Note if  $x = \sqrt{3}$ , then

$$F(x) = E(\frac{3^{3/2}}{3} - 3^{\frac{1}{2}}) = E(\underline{\frac{3^{3/2}}{3} - \frac{3^{3/2}}{3}}) = E(0) = 0$$

Therefore for  $0 < x < \sqrt{3}$ , then  $F(x) = E(\frac{x^3}{3} - x)$  is negative.

For  $x > \sqrt{3}$ , then  $F(x)$  is positive. In fact,  $F(x)$  is monotonic increasing and  $F(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

3. Since  $g(-x) = -x = -g(x)$ , it follows that the function  $g$  is odd. Also since  $g'(x) = -1$ , then the derivative of  $g$  is continuous for all  $x$ . It is obviously true that  $g(x) > 0$  for  $x > 0$ .

4. Clearly  $G(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Thus the Van der Pol equation satisfies all the properties of the Lienard-Levinson-Smith Theorem and therefore has a unique nontrivial periodic solution. In other words, the equivalent autonomous system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = E(1-x^2)y - x$$

has a unique closed path. Reducing the above autonomous system to a differential equation of the paths we have

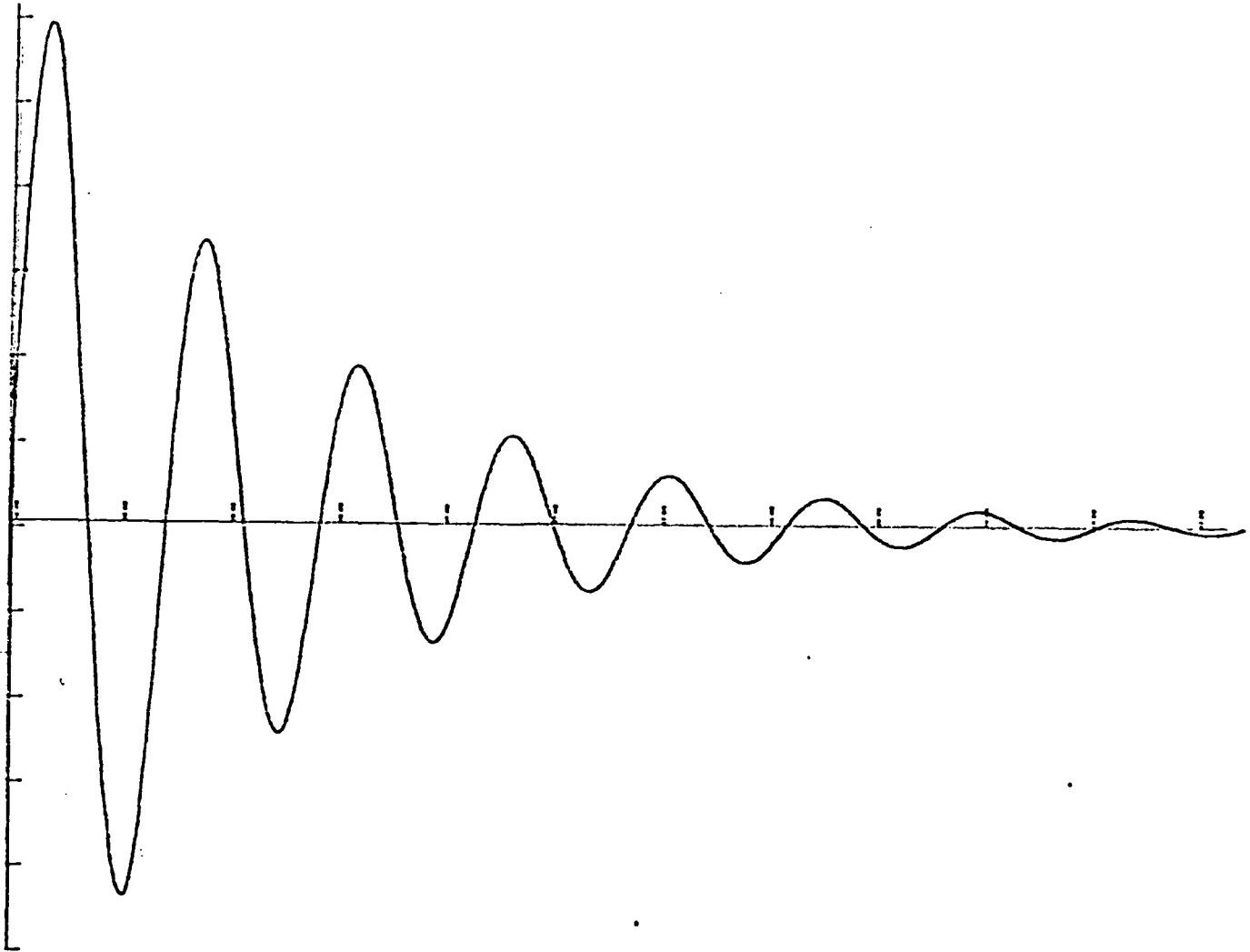
$$\frac{dy}{dt} \frac{dx}{dt} = [E(1-x^2)y - x] \cdot y$$

Using the chain rule

$$\frac{dy}{dt} \frac{dt}{dx} = [E(1-x^2)y - x] \frac{1}{y}$$

$$\frac{dy}{dx} = \frac{E(1-x^2)y - x}{y}$$

Graphs of the Van der Pol equation appear differently than graphs of equations that satisfy the Sturm Comparison Theorem. However, they depict essentially the same meaning. A graph of an equation that satisfy the Sturm Comparison Theorem appear as in the following figure.



The equation oscillates about the x-axis and reaches its limit about that axis. Graphs of the Van der Pol equation approach a limit cycle. This cycle is unique whether the approach is made from inside the closed curve or outside. Graphs of equations that satisfy the Sturm Comparison Theorem will resemble graphs of the Van der Pol equations if one would stand at the end of the x-axis and look down the x-axis toward the origin. In this

( perspective, one could observe the path taken to reach the limit cycle.

The following are graphs of the Van der Pol equation with different values of E. The closer the value of E is to zero the more the graph approximates a circle. Figure 1 is a graph with  $E=.002$ . Even though this graph is not a perfect circle, it is closer than any of the others. Figure 2 shows how the limit cycle is reached from inside and outside the closed curve with  $E=.25$ . The further away the value of E is from zero the more the closed curve becomes oblonged about the y-axis. Figure 3 also shows the limit cycle from inside and outside the closed curve with  $E=1.5$ . Figure 4 shows the limit cycle with  $E=4$ . Note how the graph is becoming more oblonged and the values of y are increasing. Figure 5 shows a more oblonged graph with  $E=7$ . Figure 6 shows the differences in the limit cycles with all the graphs starting from the same point,  $x=1.5$ , and different values of E.

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VAN DER POL EQUATION  
 $y'' + E(x^{**2} - 1)y' + y = 0$

$$E=0.002$$

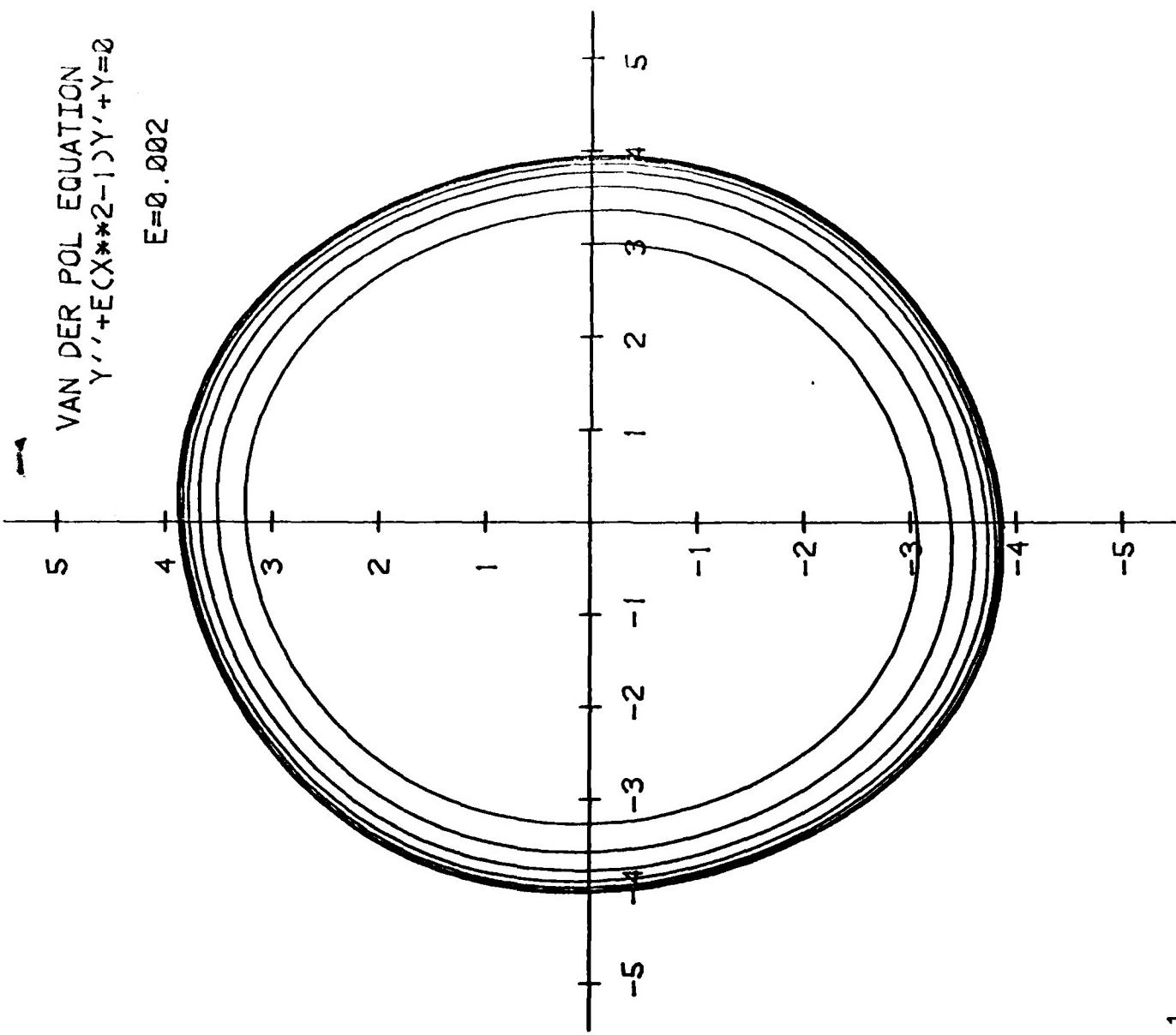


Figure 1

VAN DER POL EQUATION  
 $y_{xx} + E(x^2 - 1)y' + y = 0$

$$E=0.25$$

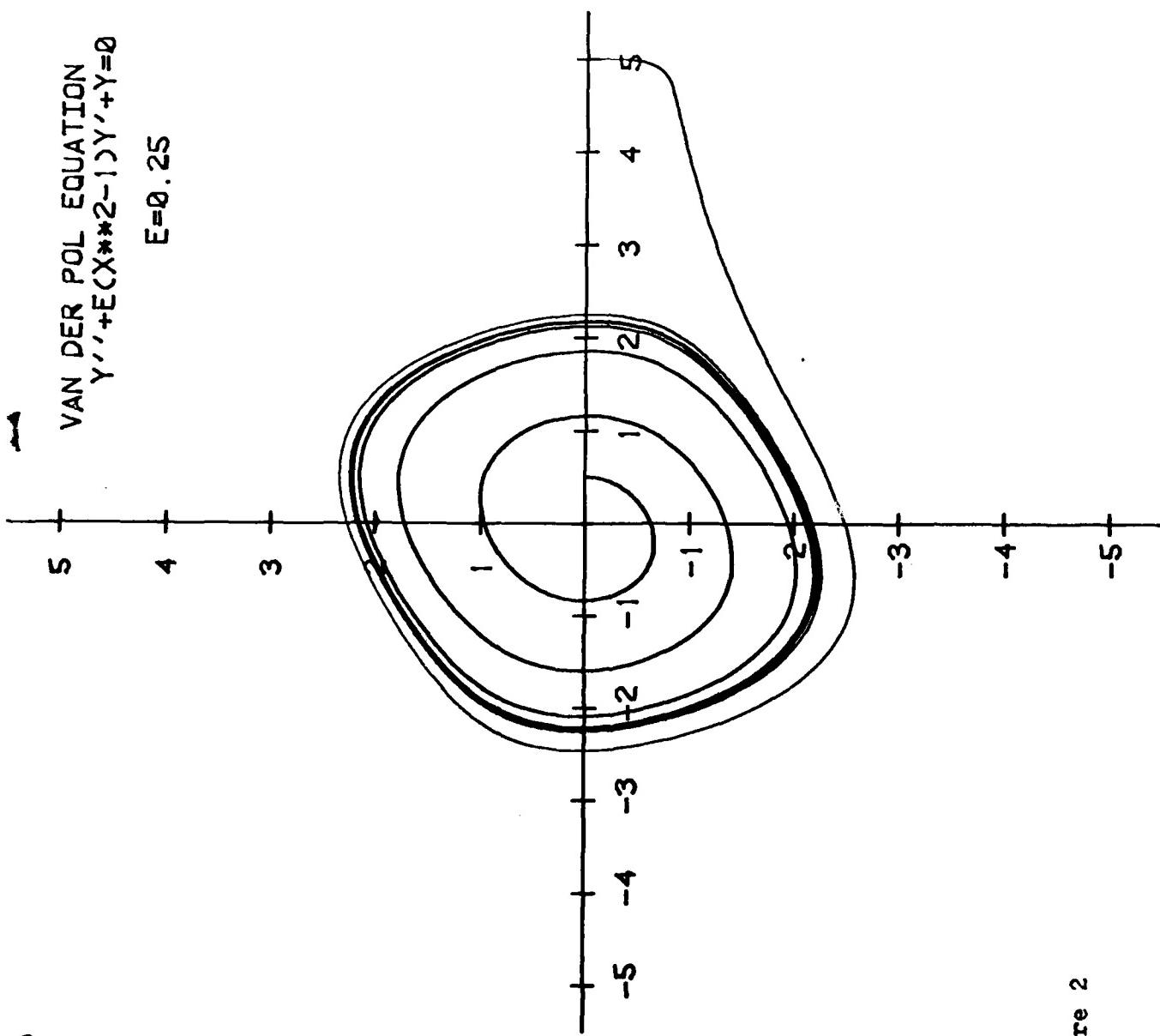


Figure 2

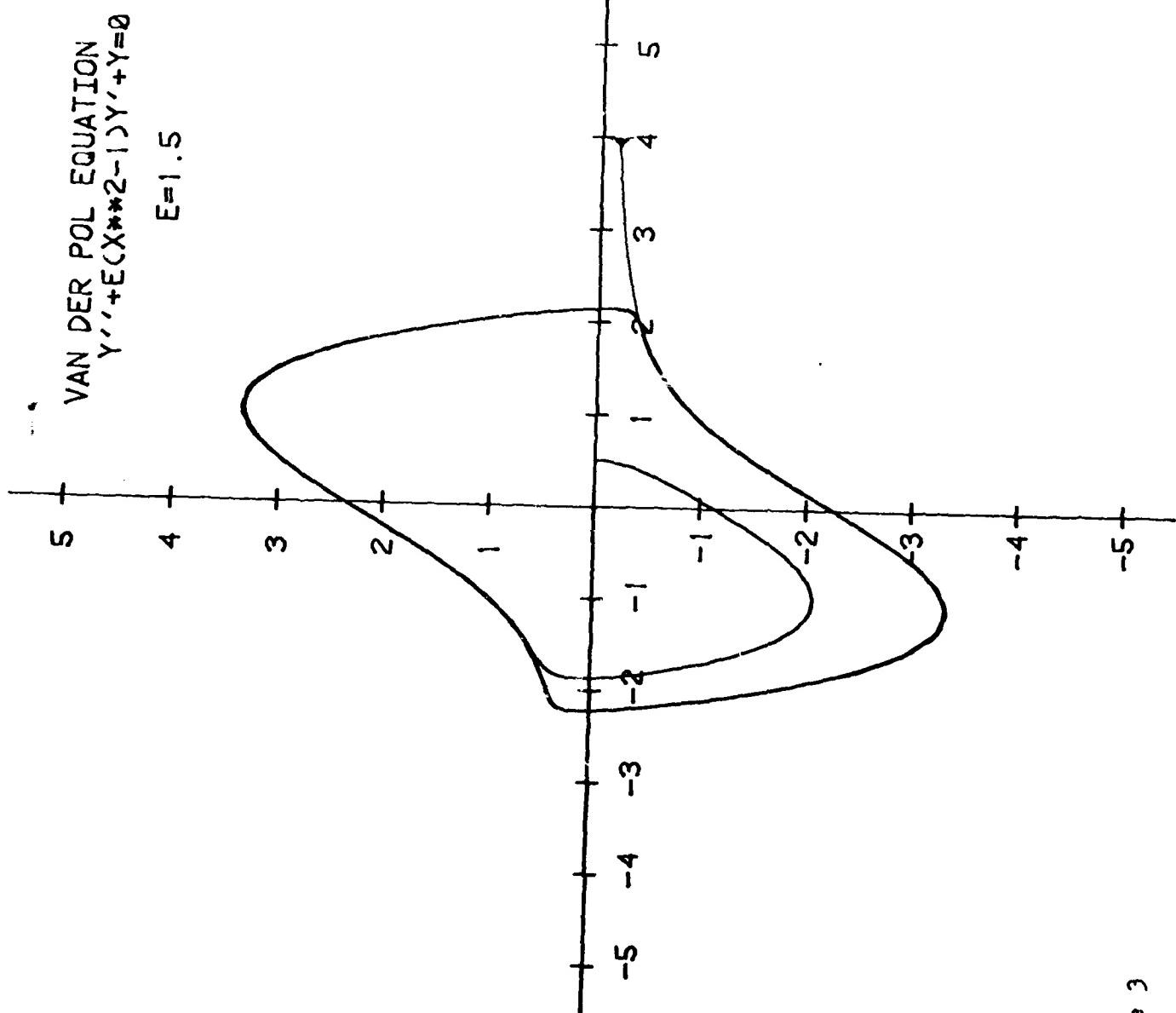


Figure 3

VAN DER POL EQUATION

$$Y'' + E(X^{**2} - 1)Y' + Y = 0$$

$$E=4$$

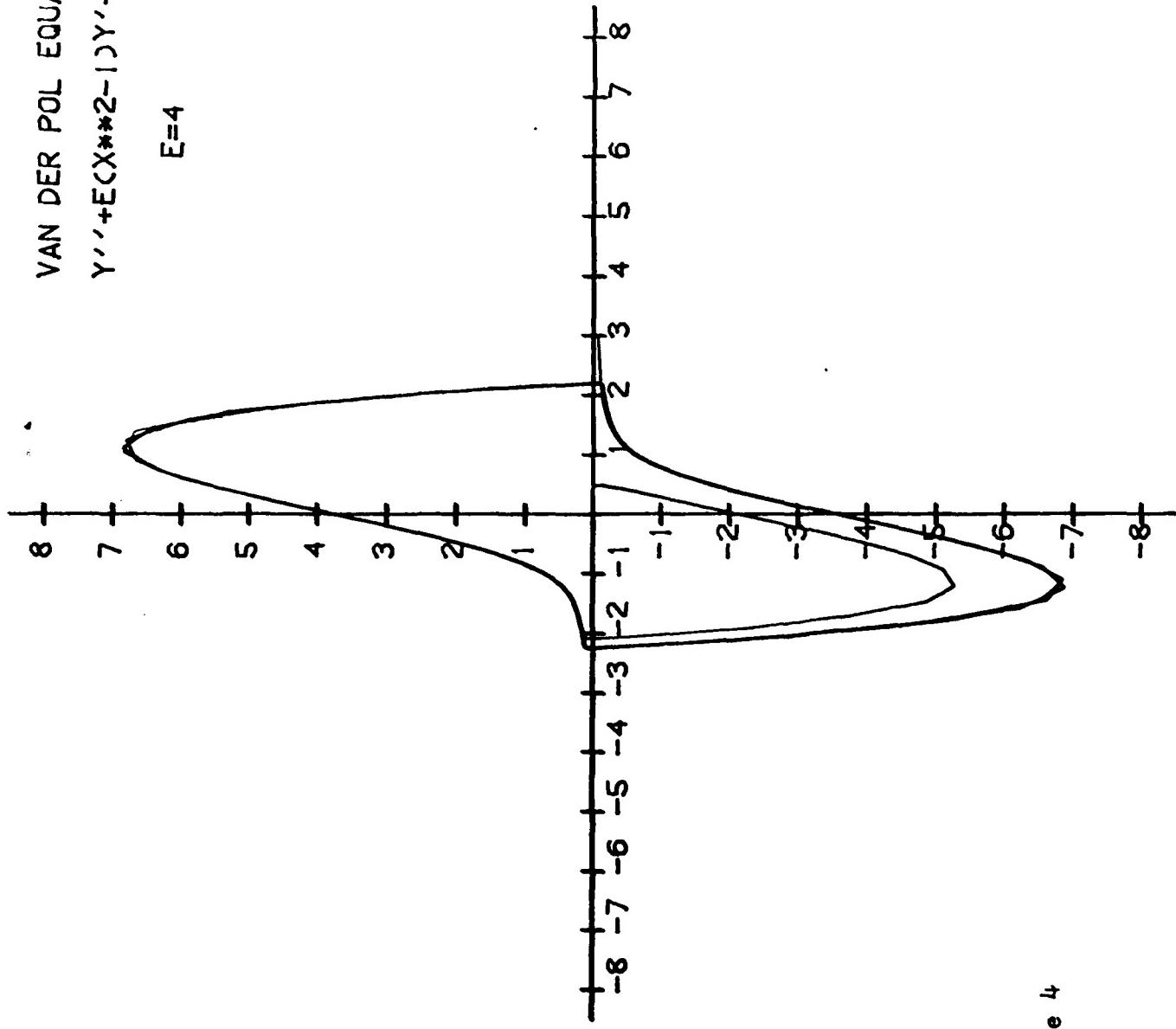


Figure 4

VAN DER POL EQUATION  
 $y'' + E(x^{*2} - 1)y' + y = 0$

$E=7$

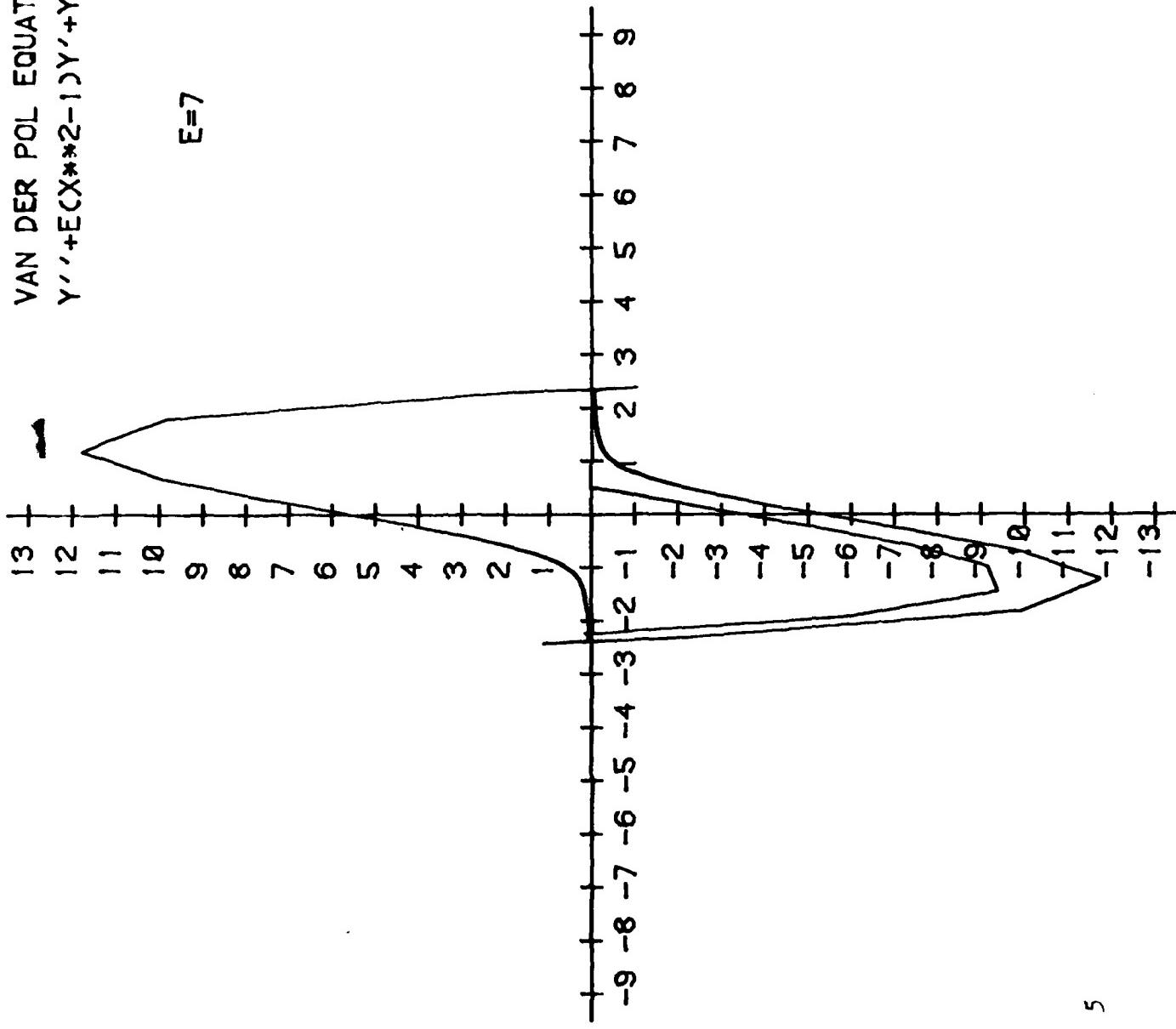


Figure 5

VAN DER POL EQUATION

$$Y'' + E(X^{**2} - 1)Y' + Y = 0$$

$$E=0.1$$

$$E=1$$

$$E=3$$

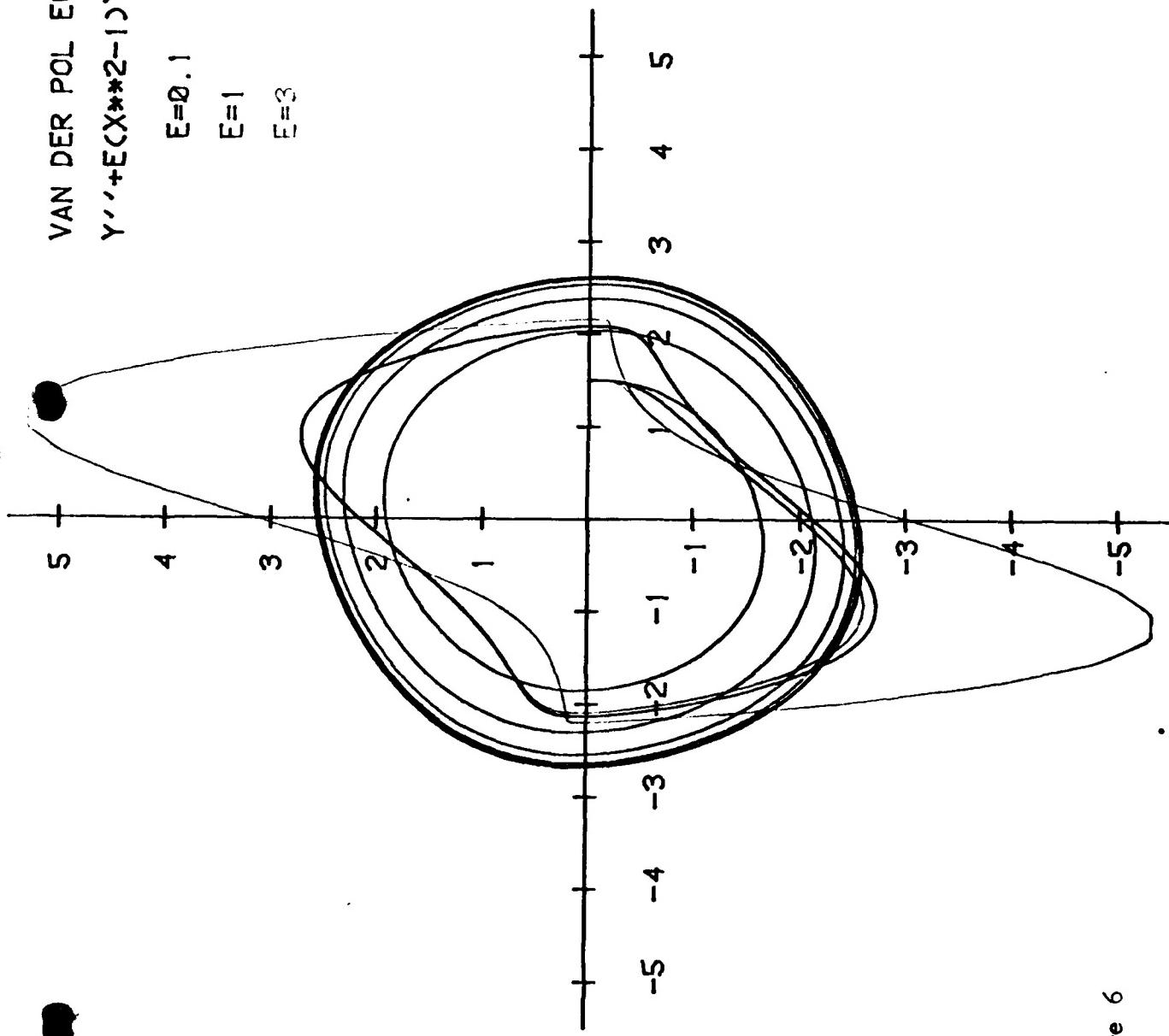


Figure 6

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